THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4010 Functional Analysis 2022-23 Term 1

Homework 8

Deadline: 2022-11-10 Thursday

Notice:

- All the assignments must be submitted before the deadline.
- Each assignment should include your name and student ID number.

1. (a) Let E_1 and E_2 be subspaces of an inner product space. Prove that $E_1 \perp E_2$ if and only if

$$\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$$

whenever $x_1 \in E_1, x_2 \in E_2$.

   (b) In contrast to part (a), give an example of a Hilbert space H and vectors $x_1, x_2 \in H$ such that $\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$, but $\langle x_1, x_2 \rangle \neq 0$.

2. Let S be a bounded sesquilinear form on $X \times Y$. Define

$$\|S\| := \sup \{|S(x, y)| : \|x\| = 1,\ \|y\| = 1\}.$$

Show that

$$\|S\| = \sup \left\{ \frac{|S(x, y)|}{\|x\| \|y\|} : x \in X \setminus \{0\},\ y \in Y \setminus \{0\} \right\}$$

and

$$|S(x, y)| \leq \|S\| \|x\| \|y\|,$$

for all $x \in X$ and $y \in Y$.

3. Let $T \colon \ell^2 \to \ell^2$ be defined by

$$T \colon (x(1), \ldots, x(n), \ldots) \mapsto (x(1), \ldots, \frac{1}{n} x(n), \ldots)$$

for $x = (x(i)) \in \ell^2$. Show that the range $\mathcal{R}(T)$ is not closed in ℓ^2.

— *THE END* —