

THE CHINESE UNIVERSITY OF HONG KONG

Department of Mathematics

MATH4010 Functional Analysis 2022-23 Term 1

Homework 5

Deadline: 2022-10-20 Thursday

Notice:

- All the assignments must be submitted before the deadline.
- Each assignment should include your name and student ID number.

1. We say that two non-empty subsets  $A$  and  $B$  of a vector space  $X$  may be *separated by a hyperplane* if there is a linear functional  $f: X \rightarrow \mathbb{R}$  and  $c \in \mathbb{R}$  such that

$$f(x) < c \text{ for } x \in A \quad \text{and} \quad f(x) > c \text{ for } x \in B.$$

Let  $c_{00}$  denote the space of finite real sequences, that is

$$c_{00} = \{(x(i)) \in \mathbb{R}^{\mathbb{N}} : \text{there exists } n \in \mathbb{N} \text{ such that } x(i) = 0 \text{ for all } i > n\}.$$

Let  $M \subset c_{00}$  be the set of sequences whose leading nonzero term is positive, that is

$$M := \{(x(i)) \in c_{00} : \text{there exists } n \in \mathbb{N} \text{ such that } x(n) > 0 \text{ and } x(i) = 0 \text{ for all } i < n\}.$$

Show that the sets  $M$  and  $-M$  are convex and disjoint, but they cannot be separated by a hyperplane. (Recall  $c_{00}^* = \ell_1$ .)

2. Let  $X$  and  $Y$  be Banach spaces and  $T: X \rightarrow Y$  a one-to-one bounded linear operator. Show that  $T^{-1}: T(X) \rightarrow X$  is bounded if and only if  $T(X)$  is closed in  $Y$ .

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