THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2068 Mathematical Analysis II (Spring 2023)
Suggested Solution of Homework 5

Q2

For any partition P of $[0,1]$, by density of $\mathbb{Q}^c$, $\inf\{h(x) : x \in [x_{k-1}, x_k]\} = 0$ for any k. Then $L(h;P) = 0$ for any partition P. Therefore, $L(h) = 0$. By the density of $\mathbb{Q}$, $\sup\{h(x) : x \in [x_{k-1}, x_k]\} \geq 1$ for any k. Then $U(h;P) \geq 1$ for any partition P. Therefore, $U(h) \geq 1$. Since $L(h;P) \neq U(h:P)$, h is not Riemann integrable on $[0,1]$.

Q8

Suppose not, i.e., there exists some $x \in [a,b]$ such that $f(x) > 0$. Since f is continuous at x, there exists some $\delta > 0$ such that $(x-\delta, x+\delta) \subset [a,b]$ and $|f(y) - f(x)| < \frac{f(x)}{2}$ for any $y \in (x-\delta, x+\delta)$. Then $f(y) > \frac{f(x)}{2}$ for any $y \in (x-\delta, x+\delta)$. Thus, $f \geq \frac{f(x)}{2}\chi_{(x-\delta,x+\delta)}$, which implies $\int_a^b f \geq \int_a^b \frac{f(x)}{2}\chi_{(x-\delta,x+\delta)} = \delta f(x) > 0$. Contradiction!

Q12

Fix $\epsilon > 0$. By Archimedean Property, there exists some $n \in \mathbb{N}$ such that $\frac{2}{(2n-1/2)\pi} < \epsilon$. Let $f(x) = 1$ for $x \in [0, \frac{1}{(2n+1/2)\pi}]$ and $f(x) = \sin(\frac{1}{x})$ for $x \in [\frac{1}{(2n+1/2)\pi}, 1]$. Then f is continuous on $[0,1]$ and $f \geq g$. Similarly, let $h(x) = -1$ for $x \in [0, \frac{1}{(2n-1/2)\pi}]$ and $h(x) = \sin(\frac{1}{x})$ for $x \in [\frac{1}{(2n-1/2)\pi}, 1]$. Then h is continuous on $[0,1]$ and $g \geq h$. Since f, h are continuous on $[0,1]$, they are Riemann integrable on $[0,1]$. Moreover, $h \leq g \leq f$ and $\int_0^1 (f-h) \leq \frac{2}{(2n-1/2)\pi} < \epsilon$. By Squeeze Theorem, g is Riemann integrable on $[0,1]$.