THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2068 Mathematical Analysis II (Spring 2023)
Suggested Solution of Homework 4

Q11

Let $f(x) = \ln x$. By induction, one can show that $f^{(n)}(x) = \frac{(-1)^{n-1}}{(n-1)!(x+1)^n}$ for any $n \in \mathbb{N}$. By Taylor's Theorem, for any $x \in (0, 1]$ and $n \in \mathbb{N}$, there exists some $c \in (0, x)$ such that $\ln x = f(x) = f(0) + f'(0)x + \cdot + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = x + \cdot + (-1)^{n-1}\frac{x^n}{n} + \frac{(-1)^n}{(c+1)^n}\frac{x^{n+1}}{n+1}$. Therefore, $|\ln x - (x + \cdot + (-1)^{n-1}\frac{x^n}{n})| = \frac{x^{n+1}}{(n+1)(c+1)^n} < \frac{x^{n+1}}{n+1}$. Taking $x = 0.5$ and $n = 7$ gives us the approximation of $\ln 1.5$ wuth error less than 0.001.

Q17

The tangent line to the graph at $(c, f(c))$ is given by $(x, f(c) + f'(c)(x - c))$. By Taylor's Theorem, for any $x \in I$, there exists some y between c and x such that $f(x) - [f(c) + f'(c)(x - c)] = \frac{f''(y)}{2}(x - c)^2$. By our assumption, $f''(y) \geq 0$. Thus $f(x) - [f(c) + f'(c)(x - c)] \geq 0$, i.e., the graph of f on I is never below the tangent line to the graph at $(c, f(c))$.