THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2068 Mathematical Analysis II (Spring 2023)
Suggested Solution of Homework 3

Q4

Let $f(x) := \sqrt{x+1}$. Then $f'(x) = \frac{1}{2}(x+1)^{-1/2}$, $f''(x) = -\frac{1}{4}(x+1)^{-3/2}$ and $f'''(x) = \frac{3}{8}(x+1)^{-5/2}$. By Taylor's Theorem, for any $x > 0$, there exists some $c \in (0, x)$ such that $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(c)}{3!}x^3 = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}(c+1)^{-5/2} > 1 + \frac{x}{2} - \frac{x^2}{8}$. Again by Taylor's Theorem, for any $x > 0$, there exists some $d \in (0, x)$ such that $f(x) = f(0) + f'(0)x + \frac{f''(d)}{2!}x^2 = 1 + \frac{x}{2} - \frac{x^2}{8}(d+1)^{-3/2} < 1 + \frac{x}{2}$.

Q8

Fix x_0, x with $x_0 < x$. By Taylor's Theorem, for any $n \in \mathbb{N}$, there exits some $c_n \in (x_0, x)$ such that $R_n = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-x_0)^{n+1} = \frac{e^{c_n}}{(n+1)!}(x-x_0)^{n+1}$. Then $\frac{R_{n+1}}{R_n} = \frac{e^{c_{n+1}-c_n}}{n+1}(x-x_0) \in (\frac{e^{x_0-x}}{n+1}(x-x_0), \frac{e^{x-x_0}}{n+1}(x-x_0))$. By Squeeze Theorem, $\lim_{n\to\infty} \frac{R_{n+1}}{R_n} = 0$. By Theoreom 3.2.11, $\lim_{n\to\infty} R_n = 0$.

Q10

Claim: $h^{(n)}(x) = e^{-1/x^2}P_n(\frac{1}{x})$ where P_n are polynomials of degree less than 3n. We shall prove this by induction. When $n = 1$, $h'(x) = e^{-1/x^2}\frac{2}{x^3}$. Suppose the statement holds for $n = k$. When $n = k+1$, $h^{(k+1)}(x) = (h^k(x))' = (e^{-1/x^2}P_k(\frac{1}{x}))' = e^{-1/x^2}(\frac{2}{x^3}P_k(\frac{1}{x}) + P_k'(\frac{1}{x})) = e^{-1/x^2}P_{k+1}(\frac{1}{x})$ where $P_{k+1}(x) = 2x^3P_k(x) + P_k'(x)$.

To show $h^{(n)}(0) = 0$, it suffices to show $\lim_{x\to 0}\frac{e^{-1/x^2}}{x^n} = 0$ for any $n \in \mathbb{N}$. We shall prove this by induction. When $n = 0$, $\lim_{x\to 0} e^{-1/x^2} = 0$. Suppose the statement holds for $n \leq k$. When $n = k+1$, $\lim_{x\to 0}\frac{e^{-1/x^2}}{x^{k+1}} = \lim_{y\to\infty}\frac{y^{(k+1)/2}}{e^y} = \frac{k+1}{2}\lim_{x\to\infty}\frac{y^{(k-1)/2}}{e^y} = \frac{k+1}{2}\lim_{x\to 0}\frac{e^{-1/x^2}}{x^{k-1}} = 0$.
Since $h^{(k)}(0) = 0$ for any $k \in \mathbb{N}$, $P_n(x) = 0$ for any $n \in \mathbb{N}$. But $f(x) \neq 0$. Hence, $R_n(x)$ does not converge to 0.