THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2068 Mathematical Analysis II (Spring 2023)
Suggested Solution of Homework 2

Q5

Let $f(x) := x^{1/n} - (x-1)^{1/n}$. Then $f'(x) = \frac{1}{n}(x^{1/n-1} - (x-1)^{1/n-1})$. Since $n \geq 2$, then $\frac{1}{n} - 1 < 0$. For any $x > 1$, $x > x - 1 > 0$. Thus $x^{1/n-1} < (x-1)^{1/n-1}$, i.e., $f'(x) < 0$. Therefore, f is decreasing for $x \geq 1$. Since $a > b > 0$, then $\frac{a}{b} > 1$. Hence, $\frac{a}{b}^{1/n} - (\frac{a}{b} - 1)^{1/n} = f(\frac{a}{b}) < f(1) = 1$, i.e., $a^{1/n} - b^{1/n} < (a-b)^{1/n}$.

Q7

Let $f(x) := \ln x$. Then $f'(x) = \frac{1}{x}$. By Mean Value Theorem, for any $x > 1$, there exists some $c \in (1, x)$ such that $\ln x = \ln x - \ln 1 = f(x) - f(1) = f'(c)(x-1) = \frac{1}{c}(x-1)$. Since $1 < c < x$, then $\frac{x-1}{x} < \frac{1}{c}(x-1) < x - 1$. Hence, $\frac{x-1}{x} < \ln x < x - 1$ for any $x > 1$.

Q8

Fix $\epsilon > 0$. Since $\lim_{x \to a} f'(x) = A$, there exists some $\delta > 0$ such that $|f'(y) - A| < \epsilon$ for any $y \in I$ satisfying $a < y < a + \delta$. For any $x \in I$ satisfying $a < x < a + \delta$, by Mean Value Theorem, there exists some $c \in (a, x)$ such that $\frac{f(x)-f(a)}{x-a} = f'(c)$. Since $a < c < x < a + \delta$, $|f'(c) - A| < \epsilon$. Therefore, $|\frac{f(x)-f(a)}{x-a} - A| < \epsilon$. Hence, $f'(a) = A$.