THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2060 Mathematical Analysis II (Spring 2023)
Suggested Solution of Homework 4

Section 7.1

5. Let $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ be a tagged partition of $[a,b]$ and let $c_1 < c_2$.

   (a) If u belongs to a subinterval I_i whose tag satisfies $c_1 \le t_i \le c_2$, show that $c_1 - \|\dot{\mathcal{P}}\| \le u \le c_2 + \|\dot{\mathcal{P}}\|$.

   (b) If $v \in [a,b]$ and satisfies $c_1 + \|\dot{\mathcal{P}}\| \le v \le c_2 - \|\dot{\mathcal{P}}\|$, then the tag t_i of any subinterval I_i that contains v satisfies $t_i \in [c_1, c_2]$.

Solution. (a) Write $I_i = [x_{i-1}, x_i]$. Then $x_{i-1} \le u, t_i \le x_i$, and hence

$$t_i - (x_i - x_{i-1}) = x_{i-1} - (x_i - t_i) \le u \le x_i + (t_i - x_{i-1}) = t_i + (x_i - x_{i-1}).$$

Since $c_1 \le t_i \le c_2$ and $0 < x_i - x_{i-1} \le \|\dot{\mathcal{P}}\|$, we have $c_1 - \|\dot{\mathcal{P}}\| \le u \le c_2 + \|\dot{\mathcal{P}}\|$.

(b) We can replace the tag of I_i by v without changing $\|\dot{\mathcal{P}}\|$. Then, since $t_i \in I_i$, it follows from (a) that

$$c_1 = (c_1 + \|\dot{\mathcal{P}}\|) - \|\dot{\mathcal{P}}\| \le t_i \le (c_2 - \|\dot{\mathcal{P}}\|) + \|\dot{\mathcal{P}}\| = c_2.$$

□

6. (a) Let $f(x) := 2$ if $0 \le x < 1$ and $f(x) := 1$ if $1 \le x \le 2$. Show that $f \in \mathcal{R}[0,2]$ and evaluate its integral.

   (b) Let $h(x) := 2$ if $0 \le x < 1$, $h(1) := 3$ and $h(x) := 1$ if $1 < x \le 2$. Show that $h \in \mathcal{R}[0,2]$ and evaluate its integral.

Solution. Fix $c \in \mathbb{R}$ and define $g : [0,2] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 2 & \text{if } 0 \le x < 1; \\ c & \text{if } x = 1; \\ 1 & \text{if } 1 < x \le 2. \end{cases}$$

We will show that, regardless of the value of c, we always have $g \in \mathcal{R}[0,2]$ and $\int_0^2 g = 3$.

Let $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be a tagged partition of $[0,2]$. Suppose $x_{k-1} \le 1 \le x_k$. Let $\dot{\mathcal{P}}_1 = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{k-1}$ and $\dot{\mathcal{P}}_2 = \{([x_{i-1}, x_i], t_i)\}_{i=k+1}^n$. Then we have

$$S(g; \dot{\mathcal{P}}) = S(g; \dot{\mathcal{P}}_1) + g(t_k)(x_k - x_{k-1}) + S(g; \dot{\mathcal{P}}_2),$$

where

$$S(g;\dot{\mathcal{P}}_1) = \sum_{i=1}^{k-1} g(t_i)(x_i - x_{i-1}) = 2(x_{k-1} - x_0) = 2 - 2(1 - x_{k-1}),$$

$$S(g;\dot{\mathcal{P}}_2) = \sum_{i=k+1}^{n} g(t_i)(x_i - x_{i-1}) = (x_n - x_k) = 1 - (x_k - 1).$$

Let $M = \max\{1, 2, |c|\}$. Then

$$\begin{aligned} \left|S(g;\dot{\mathcal{P}}) - 3\right| &\le 2|1 - x_{k-1}| + |g(t_k)||x_k - x_{k-1}| + |x_k - 1| \\ &\le 2\|\dot{\mathcal{P}}\| + M\|\dot{\mathcal{P}}\| + \|\dot{\mathcal{P}}\| \\ &= (3 + M)\|\dot{\mathcal{P}}\|. \end{aligned}$$

Now for any $\varepsilon > 0$, we can take $\delta := \varepsilon/(3 + M) > 0$, so that any tagged partition $\dot{\mathcal{P}}$ of $[0, 2]$ with $\|\dot{\mathcal{P}}\| < \delta$ satisfies

$$\left|S(g;\dot{\mathcal{P}}) - 3\right| < (3 + M)\delta = \varepsilon.$$

Therefore, $g \in \mathcal{R}[0, 2]$ and $\int_0^2 g = 3$. □

8. If $f \in \mathcal{R}[a, b]$ and $|f(x)| \le M$ for all $x \in [a, b]$, show that $\left|\int_a^b f\right| \le M(b - a)$.

Solution. Note that $-M \le f(x) \le M$ for all $x \in [a, b]$. By Example 7.1.4(a), a constant function $g(x) := k$ is Riemann integrable on $[a, b]$ and $\int_a^b g = k(b - a)$. It follows form Theorem 7.1.5 that

$$-M(b - a) = \int_a^b -M \le \int_a^b f \le \int_a^b M = M(b - a).$$

This is just $\left|\int_a^b f\right| \le M(b - a)$. □

10. Let $g(x) := 0$ if $x \in [0, 1]$ is rational and $g(x) := 1/x$ if $x \in [0, 1]$ is irrational. Explain why $g \notin \mathcal{R}[0, 1]$. However, show that there exists a sequence $(\dot{\mathcal{P}}_n)$ of tagged partitions of $[a, b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ and $\lim_n S(g;\dot{\mathcal{P}}_n)$ exists.

Solution. Let $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of $[a, b]$. If we choose a rational tag r_i for each subinterval $[x_{i-1}, x_i]$, then

$$S(g;\{([x_{i-1}, x_i], r_i)\}_{i=1}^n) = 0;$$

while if we choose an irrational tag q_i for each subinterval $[x_{i-1}, x_i]$, then

$$S(g;\{([x_{i-1}, x_i], q_i)\}_{i=1}^n) \ge 1.$$

Since $\|\mathcal{P}\| > 0$ can be arbitrarily small, we have for any $L \in \mathbb{R}$, there exists $\varepsilon_0 := 1/2$ such that for any $\delta > 0$, there is a tagged partition $\dot{\mathcal{P}}$ of $[a, b]$ such that $\|\dot{\mathcal{P}}\| < \delta$ and

$$\left|S(g; \dot{\mathcal{P}}) - L\right| \geq \varepsilon_0.$$

Hence $g \notin \mathcal{R}[0, 1]$.

Finally, we let $(\dot{\mathcal{P}}_n)$ be a sequence of tagged partitions of $[a, b]$ defined by $\dot{\mathcal{P}}_n = \{([\frac{i-1}{n}, \frac{i}{n}], \frac{i}{n})\}_{i=1}^n$. Then $\|\dot{\mathcal{P}}_n\| = \frac{1}{n} \to 0$ and $S(g; \dot{\mathcal{P}}_n) = 0$ for all $n \in \mathbb{N}$. □

12. Consider the Dirichlet function, introduced in Example 5.1.6(g), defined by $f(x) := 1$ for $x \in [0, 1]$ rational and $f(x) := 0$ for $x \in [0, 1]$ irrational. Use the preceding exercise to show that f is *not* Riemann integrable on $[0, 1]$.

Solution. Let $(\dot{\mathcal{P}}_n)$, $(\dot{\mathcal{Q}}_n)$ be two sequences of tagged partitions of $[a, b]$ defined by

$$\dot{\mathcal{P}}_n = \left\{\left(\left[\frac{i-1}{n}, \frac{i}{n}\right], \frac{i-1}{n}\right)\right\}_{i=1}^n, \quad \dot{\mathcal{Q}}_n = \left\{\left(\left[\frac{i-1}{n}, \frac{i}{n}\right], \frac{i-1}{n} + \frac{1}{\sqrt{2}n}\right)\right\}_{i=1}^n.$$

Then $\|\dot{\mathcal{P}}_n\| = \|\dot{\mathcal{Q}}_n\| = \frac{1}{n} \to 0$. However, $S(f; \dot{\mathcal{P}}_n) = 1$ while $S(f; \dot{\mathcal{Q}}_n) = 0$ for all $n \in \mathbb{N}$. Since $\lim_n S(f; \dot{\mathcal{P}}_n) \neq \lim_n S(f; \dot{\mathcal{Q}}_n)$, f is not Riemann integrable on $[0, 1]$ by Exercise 7.1.11. □

15. If $f \in \mathcal{R}[a, b]$ and $c \in \mathbb{R}$, we define g on $[a+c, b+c]$ by $g(y) := f(y - c)$. Prove that $g \in \mathcal{R}[a + c, b + c]$ and that $\int_{a+c}^{b+c} g = \int_a^b f$. The function g is called the c-**translate** of f.

Solution. First we observe that if $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a tagged partition of $[a + c, b + c]$, then $\dot{\mathcal{P}}_c := \{([x_{i-1} - c, x_i - c], t_i - c)\}_{i=1}^n$ is a tagged partition of $[a, b]$ and $\|\dot{\mathcal{P}}_c\| = \|\dot{\mathcal{P}}\|$.

Let $\varepsilon > 0$. Since $f \in \mathcal{R}[a, b]$, there exists $\delta > 0$ such that if $\dot{\mathcal{Q}}$ is any tagged partition of $[a, b]$ with $\|\dot{\mathcal{Q}}\| < \delta$, then

$$\left|S(f; \dot{\mathcal{Q}}) - \int_a^b f\right| < \varepsilon.$$

Now, if $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a tagged partition of $[a + c, b + c]$ with $\|\dot{\mathcal{P}}\| < \delta$, then

$$S(g; \dot{\mathcal{P}}) = \sum_{i=1}^n g(t_i)(x_i - x_{i-1}) = \sum_{i=1}^n f(t_i - c)((x_i - c) - (x_{i-1} - c)) = S(f, \dot{\mathcal{P}}_c).$$

Since $\dot{\mathcal{P}}_c$ is a tagged partition of $[a, b]$ with $\|\dot{\mathcal{P}}_c\| = \|\dot{\mathcal{P}}\| < \delta$, we have

$$\left|S(g; \dot{\mathcal{P}}) - \int_a^b f\right| = \left|S(f; \dot{\mathcal{P}}_c) - \int_a^b f\right| < \varepsilon.$$

Therefore, $g \in \mathcal{R}[a + c, b + c]$ and $\int_{a+c}^{b+c} g = \int_a^b f$. □