(cont'd)

On the other hand, $\exists\, N_1 > 0$ s.t.

$$d_H(A_{n+k}, A_n) < \varepsilon/2, \quad \forall\, n \geq N_1, \quad \forall\, k = 1, 2, 3, \dots$$

Then for any $b \in A_n$, $\exists\, b_k \in A_{n+k}$ s.t.

$$|b - b_k| < \varepsilon/2$$

Since $\{b_k\} \subset [0,1]$, $\exists$ subseq $b_{k_\ell} \to b^* \in [0,1]$.

By lemma, $b^* \in A$.

Hence for $b \in A_n$,

$$d(b, A) \leq |b - b^*| \leq |b - b_{k_\ell}| + |b_{k_\ell} - b^*|$$

$$< \frac{\varepsilon}{2} + |b_{k_\ell} - b^*|$$

Letting $\ell \to \infty$, $\quad d(b, A) \leq \frac{\varepsilon}{2}$.

Since $b \in A_n$ is arbitrary, we have

$$\sup_{b \in A_n} d(b, A) \leq \frac{\varepsilon}{2} \quad \forall\, n \geq N_1$$

Combining with $\displaystyle\sup_{a \in A} d(a, A_n) < \varepsilon$, $\forall\, n \geq N_0$, we have $\quad d_H(A, A_n) < \varepsilon, \quad \forall n \geq \max\{N_0, N_1\}$.

$\therefore \quad A_n \to A$ in $(\mathcal{K}, d_H)$. $\quad$ ※

Remark: By the proof of Prop 2. If A_n is Cauchy, then

$$\lim_{n\to\infty} A_n = A = \{a \in [0,1] : \exists\, a_n \in A_n \text{ s.t. } \lim_{n\to\infty} a_n = a\},$$

$$= A' = \left\{ a \in [0,1] : \begin{array}{l} \exists \text{ subseq } n_k \text{ and } a_{n_k} \in A_{n_k} \\ \text{s.t. } \lim_{k\to\infty} a_{n_k} = a \end{array} \right\}.$$

Prop 3: $\mathcal{S}_1 = \{A \in \mathcal{K} : A \text{ has nonempty interior}\}$ is of 1^{st} category in $(\mathcal{K}, d_H)$.

Pf: A has nonempty interior, if and only if

$$A \supset (\alpha, \beta) \quad \text{for some } (\alpha, \beta) \subset [0,1].$$

For any $\alpha < \beta \in [0,1]$, denote

$$S_{\alpha,\beta} = \{A \in \mathcal{K} : A \supset (\alpha, \beta)\}$$

Claim 1 : $S_{\alpha,\beta}$ is closed.

Pf : Let A_n be a seq. in $S_{\alpha,\beta}$ s.t.

$$A_n \to A \text{ in } (\mathcal{K}, d_H)$$

By the proof of Prop 2 (see the remark),

$$A = \{a \in [0,1] : \exists\, a_n \in A_n \text{ with } \lim_{n\to\infty} a_n = a\}$$

Now $\forall a \in (\alpha,\beta)$, $a_n = a \in A_n$, $\forall n = 1,2,3,\dots$

$$\Rightarrow \quad a = \lim_{n\to\infty} a_n \in A.$$

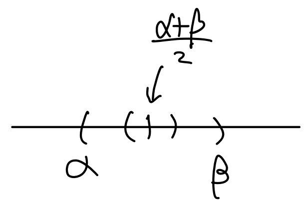
$$\Rightarrow \quad A \supset (\alpha,\beta) \text{ \& hence } A \in S_{\alpha,\beta}. \quad ※$$

Claim 2 : $\mathcal{K} \setminus S_{\alpha,\beta}$ is dense in $(\mathcal{K}, d_H)$.

Pf : Only need to show that $\forall A \in S_{\alpha,\beta}$ & $\forall \varepsilon > 0$ (suff. small)

$$B^H_\varepsilon(A) \cap (\mathcal{K} \setminus S_{\alpha,\beta}) \neq \emptyset.$$

Let $A' = A \setminus \left(\frac{\alpha+\beta}{2} - \frac{\varepsilon}{2}, \frac{\alpha+\beta}{2} + \frac{\varepsilon}{2}\right)$



A is closed implies A' is closed.

$\therefore A' \in \mathcal{K}$.

Since A' doesn't contain $\left(\frac{\alpha+\beta}{2}-\frac{\varepsilon}{2}, \frac{\alpha+\beta}{2}+\frac{\varepsilon}{2}\right)$,

$$A' \notin S_{\alpha,\beta}$$

$$\therefore \quad A' \in \mathcal{K} \setminus S_{\alpha,\beta} .$$

Since $A' \subset A$, $\forall a' \in A'$, $d(a', A) = 0$

$$\Rightarrow \quad \sup_{a' \in A'} d(a', A) = 0 .$$

On the other hand, for $a \in A$,

if $a \notin \left(\frac{\alpha+\beta}{2}-\frac{\varepsilon}{2}, \frac{\alpha+\beta}{2}+\frac{\varepsilon}{2}\right)$, then $a \in A'$

$$\Rightarrow \quad d(a, A') = 0$$

if $a \in \left(\frac{\alpha+\beta}{2}-\frac{\varepsilon}{2}, \frac{\alpha+\beta}{2}+\frac{\varepsilon}{2}\right)$, $\exists\, a' \in A'$

such that $\quad |a - a'| \leq \varepsilon/2$

$$\Rightarrow \quad d(a, A') = \inf_{a' \in A'} |a - a'| \leq \varepsilon/2$$

Hence together with the previous case,

$$\sup_{a \in A} d(a, A') < \varepsilon$$

$$\therefore \quad d_H(A, A') = \max\left\{ \sup_{a \in A} d(a, A'),\ \sup_{a' \in A'} d(a', A) \right\}$$

$$< \varepsilon$$

$$\Rightarrow \quad A' \in B^H_\varepsilon(A)$$

Hence $\quad A' \in B^H_\varepsilon(A) \cap (\mathcal{K} \setminus S_{\alpha,\beta})$

$$\Rightarrow \quad B^H_\varepsilon(A) \cap (\mathcal{K} \setminus S_{\alpha,\beta}) \neq \emptyset$$

$\#$

Claims 1 & 2 together $\Rightarrow \forall\ (\alpha, \beta) \subset [0,1]$

$S_{\alpha,\beta}$ is nowhere dense in $(\mathcal{K}, d_H)$

Hence $\displaystyle\bigcup_{\substack{\alpha,\beta \in \mathbb{Q} \\ \alpha < \beta}} S_{\alpha,\beta}$ is of 1^{st} Category

since $\mathbb{Q}$ is countable.

Finally, $\mathbb{Q}$ dense in $\mathbb{R} \Rightarrow \displaystyle \mathcal{S}_1 \subset \bigcup_{\substack{\alpha,\beta \in \mathbb{Q} \\ \alpha < \beta}} S_{\alpha,\beta}$

$\therefore \quad \mathcal{S}_1$ is of 1^{st} category. $\#$

Prop 4: $\mathcal{S}_2 = \{ A \in \mathcal{K} : A$ has at least one isolated point $\}$ is of 1^{st} category in $(\mathcal{K}, d_H)$.

Pf: $S_k = \left\{ A \in \mathcal{K} : \begin{array}{c} \exists\, a \in A \text{ s.t. } d(a, A\setminus\{a\}) \geq \frac{1}{k} \\ \text{or } A = \{a\} \end{array} \right\}$ $(k = 1, 2, \cdots)$

Claim 1: S_k is closed.

Pf: Let $A_n \in S_k$ and $A_n \to A$ in $(\mathcal{K}, d_H)$

By def. of S_k, $\exists\, a_n \in A_n$ s.t.

$$d(a_n, A_n \setminus \{a_n\}) \geq \frac{1}{k}, \quad \forall n$$

or $\quad A_n = \{a_n\}$.

Since $a_n \in [0,1]$, $\exists$ subseq a_{n_ℓ} s.t.

$$a_{n_\ell} \to a$$

By Lemma, $a \in A$.

If $A = \{a\}$, then $A \in S_k$.

If $A \neq \{a\}$, then $\exists\, b \in A \setminus \{a\}$.

By Lemma, $\exists\, b_n \in A_n$ s.t. $\lim_{n \to \infty} b_n = b$.

If $\exists$ subseq n_k s.t. $\{b_{n_k}\}$ is a subseq. of $\{a_{n_\ell}\}$

then $$b = \lim_{k\to\infty} b_{n_k} = a$$

which contradicts our choice of b.

Hence $\exists\, n_0 \geq 0$ s.t.

$$b_{n_\ell} \neq a_{n_\ell} \quad \forall\, \ell \text{ large enough s.t. } n_\ell \geq n_0 .$$

$$\Rightarrow \quad |b_{n_\ell} - a_{n_\ell}| \geq \frac{1}{k}$$

Letting $\ell \to \infty$, we have $|b-a| \geq \frac{1}{k}$.

Since $b \in A \setminus \{a\}$ is arbitrary,

$$d(a, A\setminus\{a\}) = \inf_{b\in A\setminus\{a\}} |b-a| \geq \frac{1}{k}.$$

Hence $A \in S_k$. $\divideontimes$

<u>Claim 2</u> $\mathcal{K}\setminus S_k$ is dense in $(\mathcal{K}, d_H)$.

<u>Pf</u>: Only need to show that $\forall A \in S_k$, $\forall \varepsilon > 0$

$$B^H_\varepsilon(A) \cap (\mathcal{K}\setminus S_k) \neq \emptyset$$

By def. of S_k, $\exists\, a \in A$ s.t.

$$d(a, A\setminus\{a\}) \geq \frac{1}{k}$$
$$\text{or} \quad A = \{a\}.$$

Note that it is possible to have more than one such "a" if we are in the "inequality case".

Suppose that there are infinitely many such "a's", then $\exists$ seq $a_\ell \in A$ such that $a_{\ell_1} \neq a_{\ell_2}$ if $\ell_1 \neq \ell_2$

$$d(a_\ell, A\setminus\{a_\ell\}) \geq \frac{1}{k}, \quad \forall \ell \quad \text{—— } (*)$$

Since $a_\ell \in A \subset [0,1]$, $\exists$ convergence subseq a_{ℓ_j} which implies $\{a_{\ell_j}\}$ is Cauchy.

However $(*) \Rightarrow d(a_{\ell_{j_1}}, a_{\ell_{j_2}}) \geq \frac{1}{k}$, $\forall j_1, j_2$

which is a contradiction.

$\therefore$ There are at most finitely many $\{a_1, a_2, \cdots, a_N\}$ s.t.

$$d(a_j, A\setminus\{a_j\}) \geq \frac{1}{k}.$$

For any $\varepsilon > 0$, let $\delta = \min\left\{\frac{\varepsilon}{2}, \frac{1}{2k}\right\}$, and

$$A' = \left\{A \cup \left(\bigcup_{j=1}^{N} [a_j - \delta, a_j + \delta]\right)\right\} \cap [0,1]$$

Then A closed $\Rightarrow$ A' is closed $\therefore A' \in \mathcal{K}$.

Clearly $A \subset A' \Rightarrow \sup_{a \in A} d(a, A') = 0$.

On the other hand,

$$a' \in A' = \left\{A \cup \left(\bigcup_{j=1}^{N} [a_j - \delta, a_j + \delta]\right)\right\} \cap [0,1]$$

$$\Rightarrow \begin{cases} a' \in A & \Rightarrow d(a', A) = 0 \\ a' \in [a_j - \delta, a_j + \delta] & \Rightarrow d(a', A) \leq \delta \leq \frac{\varepsilon}{2} \end{cases}$$

Hence $\sup_{a' \in A'} d(a', A) \leq \varepsilon$

$$\therefore d_H(A, A') = \max\left\{\sup_{a \in A} d(a, A'), \sup_{a' \in A'} d(a', A)\right\}$$

$$< \varepsilon$$

$\therefore A' \in B_\varepsilon^H(A)$.

Clearly, the argument works for the case $A = \{a\}$

Now suppose that $\exists\, a' \in A'$ s.t.

$$A' = \{a'\} \quad \text{or} \quad d(a', A' \setminus \{a'\}) \geq \frac{1}{k}$$

If $A' = \{a'\}$, then $A \subset A' \Rightarrow A = \{a'\}$

$$\Rightarrow A' = [a' - \delta, a' + \delta] \neq \{a'\}$$

which is a contradiction

If $d(a', A' \setminus \{a'\}) \geq \frac{1}{k}$.

Then $a' \notin [a_j - \delta, a_j + \delta]$

$\Rightarrow$ $a' \in A$ and

$$d(a', A \setminus \{a'\}) \geq d(a', A' \setminus \{a'\}) \geq \frac{1}{k}$$

$\Rightarrow$ $a' = a_j$ for some j,

which is a contradiction,

$\therefore\ A' \notin S_k$

Hence $A' \in B_\varepsilon^H(A) \cap (\mathcal{K} \setminus S_k)$. $\#$

Claims 1 & 2 $\Rightarrow$ S_k is nowhere dense for all k

$\Rightarrow$ $\bigcup_{k=1}^{\infty} S_k$ is of 1^{st} category.

Since $A \in \mathcal{S}_2 \Rightarrow \exists a \in A$ & $\varepsilon > 0$ s.t.

$$A \cap B_\varepsilon(a) = \{a\} \qquad \text{(Euclidean ball)}$$

$$\Rightarrow \exists k \left(\tfrac{1}{k} < \varepsilon\right) \text{ s.t. } A \cap B_{\frac{1}{k}}(a) = \{a\}$$

$$\Rightarrow d(a, A \setminus \{a\}) \geq \tfrac{1}{k}$$

$$\therefore \mathcal{S}_2 \subset \bigcup_{k=1}^{\infty} S_k$$

Hence $\mathcal{S}_2$ is of 1^{st} category. $\#$

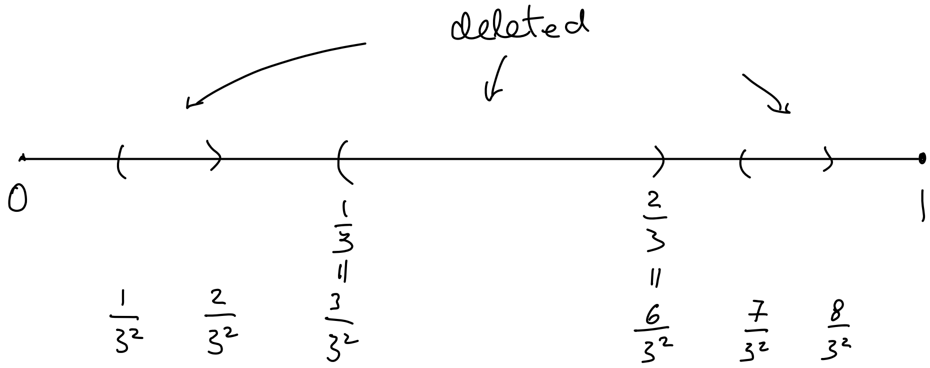
<u>Thm</u>: $\mathcal{S} = \{A \in \mathcal{K} : A$ nowhere dense without isolated pt$\}$ is a residual in $(\mathcal{K}, d_H)$ and hence dense.

<u>Pf</u>: $\mathcal{K} \setminus \mathcal{S} = \{A \in \mathcal{K} :$ A has nonempty interior or A has isolated point(s)$\}$

$$\subset \mathcal{S}_1 \cup \mathcal{S}_2.$$

Since $\mathcal{S}_1, \mathcal{S}_2$ are of 1^{st} category, $\mathcal{S}_1 \cup \mathcal{S}_2$ and hence $\mathcal{K} \setminus \mathcal{S}$ are of 1^{st} category. $\#$

<u>Remark</u>: An explicit example in $\mathcal{S}$ is the Cantor "middle-third" set (Canta ternary)



Final Exam:

Ch1 Fourier Series

- Riemann-Lebesgue Lemma
- pointwise and uniform convergence
- Weierstrass Approximation Theorem
- L^2- convergence (mean convergence)
- Parseval's Identity

Ch2 Metric Spaces

- Basic notations
- Open and Closed Sets
- Interior, closure & boundary
- Elementary Inequalities for Functions (Young's, Hölder's, Minkowski's)

Ch3 Contraction Mapping Principle

- Completeness
- Fixed points & Contraction
- Perturbation of Identity
- Inverse Function Theorem (Implicit Function Thm)
- Picard-Lindelöf Thm (IVP in ODE)

Ch4 Space of Continuous Functions

- Ascoli's Theorem
  (equicontinuity, uniform bddness, precompact)
- Arzela's Theorem
- Cauchy-Peano Thm (IVP in ODE)
- Baire Category Thm
  (nowhere dense, 1st category, residual)
- Applications of Baire Category Thm
  (eg nowhere differentiable continuous functions & etc)