Further example

$$\mathcal{K} = \{ A : \emptyset \neq A \subset [0,1] \ \& \ A \text{ closed in } [0,1] \}$$

For $A, B \in \mathcal{K}$, define

$$d_H(A,B) = \max\left\{ \sup_{a \in A} d(a,B), \sup_{b \in B} d(b,A) \right\},$$

$$\left(\begin{array}{ll} \text{Recall that} & d(a,B) = \inf_{b \in B} |a-b| \quad (\text{dist. from } a \text{ to set } B) \\ & d(b,A) = \inf_{a \in A} |b-a| \quad (\text{dist. from } b \text{ to set } A) \end{array} \right)$$

$$\boxed{\underline{\text{Prop. 1}} \quad d_H \text{ is a metric on } \mathcal{K}.}$$

<u>Pf</u>: (i) Clearly $d_H(A,B) \geq 0$.

If $d_H(A,B) = 0$, then

$$\sup_{a \in A} d(a,B) \leq d_H(A,B) = 0$$

$$\Rightarrow \quad d(a,B) = 0, \ \forall a \in A$$

$$\Rightarrow \quad \forall a \in A, \ \forall n > 0, \ \exists b_n \in B \text{ s.t.}$$

$$|a - b_n| < \tfrac{1}{n}$$

$$\Rightarrow \quad b_n \to a \text{ in } [0,1]$$

Since B is <u>closed</u>, $a \in B$. $\therefore$ $A \subset B$.

Similarly $B \subset A$.

$\therefore \quad A = B$.

(ii) Clearly $d_H(A,B) = d_H(B,A)$ by definition

(iii) Let $A, B, C \in K$

$\forall\, a \in A,\ b \in B,\ \&\ c \in C,$

$$|a-b| \leq |a-c| + |c-b|$$

$$\Rightarrow \inf_{b \in B} |a-b| \leq |a-c| + \inf_{b\in B} |c-b|$$

i.e. $$d(a,B) \leq |a-c| + d(c,B)$$

$$\leq |a-c| + \sup_{c \in C} d(c,B)$$

$$\leq |a-c| + d_H(C,B)$$

$\forall\, n > 0,\ \exists\, c_n \in C$ s.t. $|a - c_n| < d(a, C) + \frac{1}{n}$

$$\therefore \quad d(a,B) \leq d(a,C) + \frac{1}{n} + d_H(C,B)$$

$$\leq d_H(A,C) + d_H(C,B) + \frac{1}{n}$$

Hence $$\sup_{a \in A} d(a,B) \leq d_H(A,C) + d_H(C,B) + \frac{1}{n}$$

Similarly $\sup_{b \in B} d(b, A) \le d_H(A, C) + d_H(C, B) + \frac{1}{n}$

$$\Rightarrow \quad d_H(A, B) \le d_H(A, C) + d_H(C, B) + \frac{1}{n}, \quad \forall n = 1, 2, \dots$$

Letting $n \to \infty$, we have

$$d_H(A, B) \le d_H(A, C) + d_H(C, B).$$

(i), (ii) & (iii) together $\Rightarrow$ d_H is a metric on $\mathcal{K}$. #

$$\boxed{\text{Prop 2} \quad (\mathcal{K}, d_H) \text{ is complete.}}$$

Lemma Let A_n be a seq. in $\mathcal{K}$,

$$A = \{a \in [0,1] : \exists\, a_n \in A_n \text{ s.t. } \lim_{n \to \infty} a_n = a\}, \quad \text{and}$$

$$A' = \left\{ a \in [0,1] : \begin{array}{l} \exists \text{ subseq } n_k \text{ and } a_{n_k} \in A_{n_k} \\ \text{s.t. } \lim_{k \to \infty} a_{n_k} = a \end{array} \right\}.$$

If A_n is a Cauchy seq. in $(\mathcal{K}, d_H)$, then $A = A'$.

Pf: Clearly $A \subset A'$.

To prove $A' \subset A$, we consider an $a \in A'$.

By def. of A', $\exists$ subseq n_k and $a_{n_k} \in A_{n_k}$, $\forall k$ s.t.

$$\lim_{k\to\infty} a_{n_k} = a.$$

If $a \notin A$, then

$$\exists\ \varepsilon_0 > 0 \text{ and a subseq } n_l\ (\notin \{n_k\}, \forall l) \text{ s.t.}$$

$$d(a, A_{n_l}) \geq \varepsilon_0.$$

Otherwise, $\forall\ \varepsilon_0 > 0$, $\exists\ n_0 > 0$ s.t.

$$d(a, A_n) < \varepsilon_0, \qquad \forall\ n \geq n_0\ \&\ n \notin \{n_k\}$$

$\Rightarrow$ $\forall\ n \geq n_0\ \&\ n \notin \{n_k\}$, $\exists\ z_n \in A_n$ s.t.

$$|a - z_n| < \varepsilon_0,$$

Hence combining with a_{n_k}, we have a convergent seq

$$a_n = \begin{cases} z_n, & \text{if } n \neq n_k \text{ for any } k \\ a_{n_k}, & \text{if } n = n_k \text{ for some } k \end{cases}$$

limiting to a (i.e. $\lim_{n\to\infty} a_n = a$)

$\Rightarrow$ $a \in A$ which contradicts our assumption.

$\therefore \exists\, \varepsilon_0 > 0$ and a subseq n_ℓ ($\notin \{n_k\}, \forall \ell$) s.t.

$$d(a, A_{n_\ell}) \geq \varepsilon_0 .$$

Hence $\quad \varepsilon_0 \leq |a - z_{n_\ell}|, \quad \forall\, z_{n_\ell} \in A_{n_\ell}$

$$\Rightarrow \quad \varepsilon_0 \leq |a - a_{n_k}| + |a_{n_k} - z_{n_\ell}|$$

$$\leq |a - a_{n_k}| + d(a_{n_k}, A_{n_\ell}) \quad \left(\text{by take } \inf_{z_{n_\ell} \in A_\ell}\right)$$

$$\leq |a - a_{n_k}| + d_H(A_{n_k}, A_{n_\ell})$$

As $a_{n_k} \to a$ and A_n is Cauchy,

$\exists\, k_0 > 0$ s.t. $|a - a_{n_k}| < \frac{\varepsilon_0}{4}, \quad \forall\, k \geq k_0$

and $\quad d_H(A_{n_k}, A_{n_\ell}) < \frac{\varepsilon_0}{4}, \quad \forall\, k \geq k_0$ and $\forall\, n_\ell \geq n_{k_0}$

Letting ℓ large enough s.t. $n_\ell \geq n_{k_0}$, we have

$$\varepsilon_0 \leq \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} = \frac{\varepsilon_0}{2}$$

which is also a contradiction.

$\therefore\; a \in A.$

Since $a \in A'$ is arbitrary, $A' \subset A$ & hence $A = A'$ $\#$

Pf of Prop 2: Let A_n be a Cauchy seq. in $(\mathcal{K}, d_H)$

and $A = \{ a \in [0,1] : \exists\, a_n \in A_n$ with $\lim_{n\to\infty} a_n = a \}$.

Step 1 A is closed (& hence $A \in \mathcal{K}$)

Pf: Let $a \in [0,1] \setminus A$, then

$a \neq \lim_{n\to\infty} a_n$ for any seq. a_n with $a_n \in A_n$

$\Rightarrow \exists\, \varepsilon_0 > 0$ and subseq n_k s.t. for any $a_{n_k} \in A_{n_k}$

$$|a_{n_k} - a| \geq \varepsilon_0$$

Taking inf over A_{n_k}, we have

$\exists\, \varepsilon_0 > 0$ and subseq n_k s.t.

$$d(a, A_{n_k}) \geq \varepsilon_0$$

Let $b \in (a - \frac{\varepsilon_0}{2}, a + \frac{\varepsilon_0}{2})$, then $\forall\, a_{n_k} \in A_{n_k}$

$$|a - a_{n_k}| \leq |a-b| + |b - a_{n_k}|$$

$$\Rightarrow \quad \varepsilon_0 < \frac{\varepsilon_0}{2} + |b - a_{n_k}|, \ \forall\, a_{n_k} \in A_{n_k}$$

$$\Rightarrow \quad \frac{\varepsilon_0}{2} \leq d(b, A_{n_k})$$

Hence, there is no seq a_n with $a_n \in A_n$

s.t. $$\lim_{n\to\infty} a_n = b$$

$\therefore$ $b \in [0,1] \setminus A$.

Since $b \in (a-\frac{\varepsilon_0}{2}, a+\frac{\varepsilon_0}{2})$ is arbitrary,

$$(a-\tfrac{\varepsilon_0}{2}, a+\tfrac{\varepsilon_0}{2}) \subset [0,1] \setminus A.$$

As $a \in [0,1]\setminus A$ is arbitrary, $[0,1]\setminus A$ is open

$\therefore$ A is closed.

This completes the proof of Step 1.

<u>Step 2</u> $A_n \to A$ in $(\mathcal{K}, d_H)$

<u>Pf</u>: By def. of A,

$\forall \varepsilon > 0$, $\forall a \in A$, $\exists a_n \in A_n$ & $N_0 > 0$ s.t.

$$|a - a_n| < \varepsilon, \quad \forall n \geq N_0$$

$$\Rightarrow \quad d(a, A_n) < \varepsilon, \quad \forall n \geq N_0$$

$$\Rightarrow \quad \sup_{a \in A} d(a, A_n) < \varepsilon, \quad \forall n \geq N_0$$

(to be cont'd)