(Cont'd)

<u>Claim 2</u> : S_L is nowhere dense

(By claim 1, one needs to show $C[0,1] \setminus S_L$ is dense)

<u>Pf</u>: By definition, we need to show that

$\forall \varepsilon > 0$, and $f \in C[0,1]$,

$$B_\varepsilon^\infty(f) \cap (C[0,1] \setminus S_L) \neq \phi.$$

If $f \notin S_L$, it is clear: $f \in B_\varepsilon^\infty(f)$ & $f \in C[0,1] \setminus S_L$

For $f \in S_L$, Weierstrass Approximation Theorem $\Rightarrow$

$\forall \varepsilon > 0$, $\exists$ a polynomial p such that

$$\|f - p\|_\infty < \varepsilon/2$$

Let the Lip. constant of p be L_1.

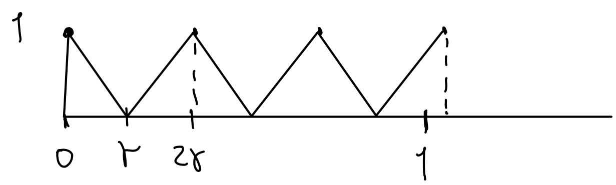
For $r > 0$ (& $r < 1$, not necessary rational), let

$\varphi(x)$ be the restriction to $[0,1]$ of the jig-saw function

of period $2r$ satisfying $\varphi(0) = 1$, $0 \le \varphi \le 1$,

and slope of the graph of φ is $\pm \frac{1}{r}$

(except the finitely many non-differentiabe points.)



Then consider the function

$$g(x) = p(x) + \frac{\varepsilon}{2}\varphi(x) \in C[0,1]$$

Then $\|g-f\|_\infty \leq \|p-f\|_\infty + \frac{\varepsilon}{2}\|\varphi\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

i.e. $\quad g \in B_\varepsilon^\infty(f)$. $\qquad (\forall\, r \in (0,1))$

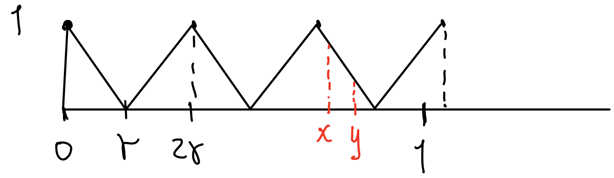
On the other hand

$$\left|\frac{\varepsilon}{2}\varphi(y) - \frac{\varepsilon}{2}\varphi(x)\right| \leq |g(y)-g(x)| + |p(y)-p(x)|$$

$$\Rightarrow \quad \frac{\varepsilon}{2}|\varphi(y)-\varphi(x)| \leq |g(y)-g(x)| + L_1|y-x|.$$

Note that $\forall\, x \in [0,1]$, $\exists\, y \in [0,1]$ near x such that

$$|\varphi(y) - \varphi(x)| = \frac{1}{r}|y-x|$$



$$\Rightarrow \quad |g(y)-g(x)| \geq \left(\frac{\varepsilon}{2r} - L_1\right)|y-x|$$

Hence if we choose $r < \frac{\varepsilon}{2(L+L_1)}$, then

$\forall x \in [0,1]$, $\exists y \in [0,1]$ such that

$$|g(y)-g(x)| \geq \left(\frac{\varepsilon}{2r} - L_1\right)|y-x| > L|y-x|.$$

i.e. $\forall x \in [0,1]$, g is not Lip. cts at x with Lip constant L.

$$\Rightarrow \quad g \notin S_L$$

$$\therefore \quad B^{\infty}_{\varepsilon}(f) \cap (C[0,1] \setminus S_L) \neq \phi$$

By claim 1, S_L is closed hence S_L is nowhere dense.

#

<u>Final Step</u>:

Let $S = \{ f \in C[0,1] : f \text{ is differentiable at some } x \in [0,1] \}$

Then by Lemma 4.12,

$\forall f \in S$, $f \in S_N$ for some $N \in \mathbb{N}$.

$$\Rightarrow \quad S \subset \bigcup_{N=1}^{\infty} S_N.$$

By claim 2, S is of 1st category.

And Baire Category Thm (using $C[0,1]$ is complete)

$\Rightarrow$ S has empty interior.

$\Rightarrow$ Set of cts, but nowhere differentiable functions on $[0,1]$

($=$ complement of S in $C[0,1]$) is a residual set

and hence dense in $C[0,1]$ $\#$

<u>Remarks</u> (i) The Thm and its proof provide no explicit example, not even a method to construct a continuous nowhere differentiable function.

(ii) An explicit example was given by Weierstrass:

$$W(x) = \sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n} \quad \text{on } \mathbb{R}$$

Further examples

Def Let
- $f : [a,b] \to \mathbb{R}$ be a function, and
- $L : \mathbb{R} \to \mathbb{R}$, $x \mapsto \alpha x + \beta$ for some $\alpha, \beta \in \mathbb{R}$ (L is degree ≤ 1 poly)

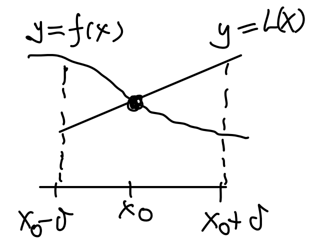
We say <u>L crosses f</u> (or <u>f crosses L</u>)

if $\exists\, x_0 \in [a,b]$, and $\delta > 0$ such that

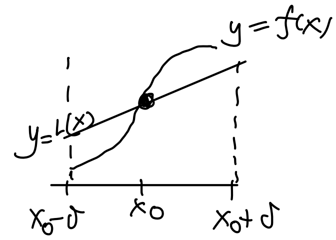
$$f(x_0) = L(x_0)$$

and either one of following holds

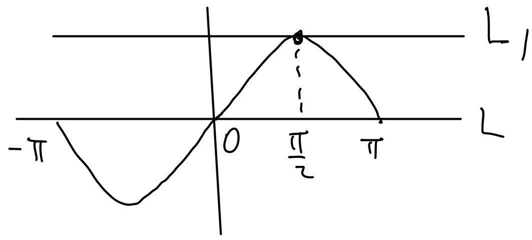
(i) $\begin{cases} L(x) \leq f(x), & \forall x \in [x_0 - \delta, x_0] \cap [a,b] \\ L(x) \geq f(x), & \forall x \in [x_0, x_0 + \delta] \cap [a,b] \end{cases}$



(ii) $\begin{cases} L(x) \geq f(x), & \forall x \in [x_0 - \delta, x_0] \cap [a,b] \\ L(x) \leq f(x), & \forall x \in [x_0, x_0 + \delta] \cap [a,b] \end{cases}$



eg $f(x) = \sin x$



- $L(x) \equiv 0$ (zero function)

Clearly L crosses f

(at $0, \pm\pi, \pm 2\pi, \cdots$)

(At $x_0 = 0$, the $\delta > 0$ can be chosen as π)

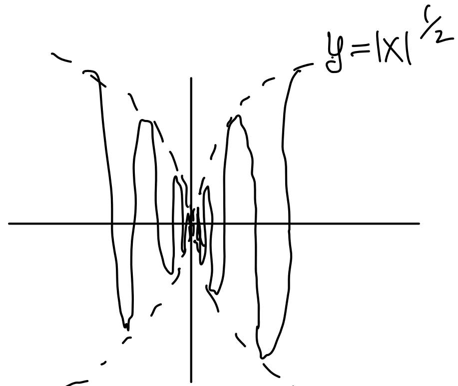
- If $L_1(x) \equiv 1$, L_1 doesn't cross f:

at every intersection $(2n+1)\frac{\pi}{2}$, for all

$$x \in \left((2n+1)\tfrac{\pi}{2} - \delta, (2n+1)\tfrac{\pi}{2} + \delta\right) \setminus \left\{(2n+1)\tfrac{\pi}{2}\right\}, \quad \forall\, 0 < \delta < 2\pi$$

$$f(x) < L_1(x) \equiv 1$$

eg: $f(x) = \begin{cases} |x|^{\frac{1}{2}} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$



Then no "line" L crosses f at $x_0 = 0$

All "line" L passing thro $(0,0)$ intersects $y = \pm|x|^{\frac{1}{2}}$.

Then infinite oscillation of $f \Rightarrow$

neither (i) nor (ii) in the definition holds.

Def: A function $f : [a,b] \to \mathbb{R}$ is said to be "crosses no lines"

if there is no $L(x) = \alpha x + \beta$ crosses f.

Thm: The set $Z = \{ f \in C[a,b] : f \text{ crosses no lines} \}$

is a residual set in $C[a,b]$, and hence dense.

Pf: Note that

$$C[a,b] \setminus Z = \{ f \in C[a,b] : \exists \text{ some } L \text{ crosses } f \text{ (at some pt.)} \}$$

$$\text{where } L(x) = \alpha x + \beta \quad (\alpha, \beta \in \mathbb{R})$$

And we need to show that $C[a,b] \setminus Z$ is of 1^{st} category.

Notation: For $f \in C[a,b]$ and $\alpha \in \mathbb{R}$, we denote

$$f_{-\alpha}(x) = f(x) - \alpha x$$

(subtracting the linear part of L from f)

Let A_n be the set of $f \in C[a,b]$ for which

$\exists\, \alpha \in [-n, n]$ and $x \in [a,b]$ such that

$$\begin{cases} f_{-\alpha}(t) \leq f_{-\alpha}(x) & \forall t \in (x-\frac{1}{n}, x) \\ f_{-\alpha}(t) \geq f_{-\alpha}(x) & \forall t \in (x, x+\frac{1}{n}) \end{cases} \qquad (t \in [a,b])$$

Clearly $A_n \subset A_{n+1}$, $\forall n$ (since $(x-\frac{1}{n+1}, x+\frac{1}{n+1}) \subset (x-\frac{1}{n}, x+\frac{1}{n})$)

Note that t is now the independent variable, and

$f_{-\alpha}(t) \leq f_{-\alpha}(x)$ is exactly

$$f(t) \leq \alpha t + (f(x) - \alpha x) \overset{\text{def}}{=} L\, t \quad , \quad \forall t \in (x-\tfrac{1}{n}, x)$$

Similarly, $f_{-\alpha}(t) \geq f_{-\alpha}(x)$ is exactly

$$f(t) \geq \alpha t + (f(x) - \alpha x) = L\, t \; , \quad \forall t \in (x, x+\tfrac{1}{n})$$

$\therefore$ $f \in A_n \Rightarrow f$ crosses L at x

(with slope $|\alpha| \leq n$)

And if f crosses some L at some x

with L given by consts α (slope) & β, then for some $\delta > 0$

either
$$\begin{cases} \alpha t + \beta \leq f(t), & \forall t \in [x-\delta, x] \cap [a,b] \\ \alpha t + \beta \geq f(t), & \forall t \in [x, x+\delta] \cap [a,b] \end{cases}$$

or $\begin{cases} \alpha t + \beta \geq f(t), & \forall\, t \in [x-\delta, x] \cap [a,b] \\ \alpha t + \beta \leq f(t), & \forall\, t \in [x, x+\delta] \cap [a,b] \end{cases}$

By continuity of $f \in C[a,b]$, we also have

$$\alpha x + \beta = f(x)$$

i.e. $$\beta = f(x) - \alpha x = f_{-\alpha}(x)$$

Hence either

$$\begin{cases} f_{-\alpha}(x) \leq f_{-\alpha}(t), & \forall\, t \in [x-\delta, x] \cap [a,b] \\ f_{-\alpha}(x) \geq f_{-\alpha}(t), & \forall\, t \in [x, x+\delta] \cap [a,b] \end{cases}$$

or $\begin{cases} f_{-\alpha}(x) \geq f_{-\alpha}(t), & \forall\, t \in [x-\delta, x] \cap [a,b] \\ f_{-\alpha}(x) \leq f_{-\alpha}(t), & \forall\, t \in [x, x+\delta] \cap [a,b] \end{cases}$

$\Rightarrow$ $-f \in A_n$ or $f \in A_n$ for some n $(n \geq |\alpha|\ \&\ \frac{1}{n} < \delta)$

$\Rightarrow$ $-f \in A = \bigcup_{n=1}^{\infty} A_n$ or $f \in A = \bigcup_{n=1}^{\infty} A_n$

Denote $B = \{ f \in C[a,b] : -f \in A \}$

Then f crosses some lines

$$\Leftrightarrow \quad f \in A \cup B$$

Hence $\quad C[a,b] \setminus Z = A \cup B$

So we only need to show that A_n is nowhere dense, $\forall n$,

by proving

(1) A_n is closed $\forall n$, and

(2) $C[a,b] \setminus A_n$ is dense.

<u>Pf of (1)</u> Let $\{f_k\}$ be a seq. in A_n and

$$f_k \to f \text{ in } (C[a,b], d_\infty)$$

Since $f_k \in A_n$, $\exists$ $\alpha_k \in [-n,n]$ and

$$x_k \in [a,b]$$

s.t. $\begin{cases} (f_k)_{-\alpha_k}(t) \leq (f_k)_{-\alpha_k}(x_k), & \forall t \in (x_k - \frac{1}{n}, x_k) \\ (f_k)_{-\alpha_k}(t) \geq (f_k)_{-\alpha_k}(x_k), & \forall t \in (x_k, x_k + \frac{1}{n}) \end{cases}$

$(t \in [a,b])$

By passing to subseq., we may assume

$$x_k \to x_0 \in [a,b]$$

$$\alpha_k \to \alpha_0 \in [-n,n]$$

Then $(f_k)_{-\alpha_k}(t) \le (f_k)_{-\alpha_k}(x_k)$, $\forall t \in (x_k - \frac{1}{n}, x_k)$

$\Leftrightarrow$ $f_k(t) - \alpha_k t \le f_k(x_k) - \alpha_k x_k$, $\forall \in (x_k - \frac{1}{n}, x_k)$

Now $\forall t \in (x_0 - \frac{1}{n}, x_0)$, $\exists k_0 \ge 0$ s.t.

$$t \in (x_k - \tfrac{1}{n}, x_k), \quad \forall k \ge k_0 \quad (\text{since } x_k \to x_0)$$

Then $f_k \to f$ in $(C[a,b], d_\infty)$, $\alpha_k \to \alpha_0$, $x_k \to x_0$

we have $f(t) - \alpha_0 t \le f(x_0) - \alpha_0 x_0$ (by letting $k \to +\infty$)

Since $t \in (x_0 - \frac{1}{n}, x_0)$ is arbitrary, we've proved

$$f_{-\alpha_0}(t) \le f_{-\alpha_0}(x_0), \quad \forall t \in (x_0 - \tfrac{1}{n}, x_0]$$

Similarly, we can prove

$$f_{-\alpha_0}(t) \ge f_{-\alpha_0}(x_0), \quad \forall t \in (x_0, x_0 + \tfrac{1}{n})$$

Hence $f \in A_n$, $\therefore$ A_n is closed.

<u>Pf of (2)</u> Let $B^\infty_\varepsilon(f) \subset C[a,b]$ be a metric ball.

If $f \notin A_n$, then $B^\infty_\varepsilon(f) \cap (C[a,b] \setminus A_n) \ne \emptyset$.

If $f \in A_n$, by Weierstrass Approximation Theorem,

$\exists$ polynomial p s.t. $\|p - f\|_\infty < \varepsilon/3$.

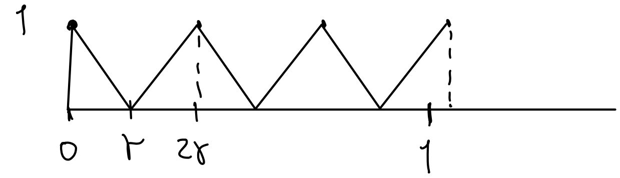
Define $\quad g(x) = p(x) + \frac{\varepsilon}{3}\varphi(x) \quad \in C[a,b]$ $\quad \left(\begin{array}{c}\text{We may assume}\\ [a,b]=[0,1]\end{array}\right)$

where φ is the restriction to $[a,b]$ of the jig-saw function

of period $2r$ satisfying $0 \le \varphi \le 1$, and

slope of the graph of φ is $\pm\frac{1}{r}$ $\quad$ ($r>0$, to be determined)

(except the finitely many non-differentiable points)



Then $\quad \|g-f\|_\infty \le \|g-p\|_\infty + \|p-f\|_\infty$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$$

$\Rightarrow \quad g \in B_\varepsilon^\infty(f)$

Suppose that $g \in A_n$

then $\exists\, x \in [a,b],\ \alpha \in [-n,n]$ s.t

$$\begin{cases} g_{-\alpha}(t) \le g_{-\alpha}(x), & t \in (x-\frac{1}{n}, x) \\ g_{-\alpha}(t) \ge g_{-\alpha}(x), & t \in (x, x+\frac{1}{n}) \end{cases}$$

If $\varphi(x) \in [0, \frac{1}{2}]$, then consider

$$g_{-\alpha}(t) \le g_{-\alpha}(x), \quad \forall\, t \in (x-\tfrac{1}{n}, x)$$

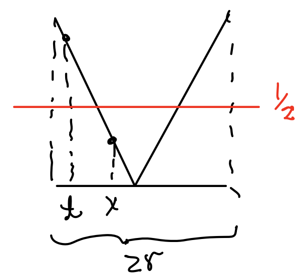
which implies $\forall\, t \in (x-\frac{1}{n}, x)$,

$$p(t) + \frac{\varepsilon}{3}\varphi(t) - \alpha t \le p(x) + \frac{\varepsilon}{3}\varphi(x) - \alpha x$$

$$\Rightarrow \quad \varphi(x) - \varphi(t) \ge \frac{3\alpha}{\varepsilon}(x-t) - \frac{3}{\varepsilon}\left(p(x) - p(t)\right)$$

By the property of φ, $\exists\, t$ with $0 < x-t < r$

s.t. $\quad \varphi(x) - \varphi(t) \le -\frac{1}{2}$



Consider $r < \min\{\frac{1}{n}, \frac{\varepsilon}{12(L+n)}\}$, where $L =$ Lip. const. of p.

By $r < \frac{1}{n}$, $t \in (x-\frac{1}{n}, x)$ and s.t.

$$-\frac{1}{2} \ge \frac{3\alpha}{\varepsilon}(x-t) - \frac{3}{\varepsilon}\left(p(x)-p(t)\right)$$

$$\Rightarrow \quad 1 \le \frac{6|\alpha|}{\varepsilon}|x-t| + \frac{6}{\varepsilon}L|x-t|$$

$$\le \frac{12(L+n)}{\varepsilon} r < 1,$$

which is a contradiction.

Hence $\varphi(x) \in [\frac{1}{2}, 1]$.

Then consider $g_{-\alpha}(t) \geq g_{-\alpha}(x)$, $\forall\, t \in (x, x+\frac{1}{n})$

which implies $\forall\, t \in (x, x+\frac{1}{n})$

$$p(t) + \frac{\varepsilon}{3}\varphi(t) - \alpha t \geq p(x) + \frac{\varepsilon}{3}\varphi(x) - \alpha x$$

$$\Rightarrow \quad \varphi(t) - \varphi(x) \geq \frac{3\alpha}{\varepsilon}(t-x) - \frac{3}{\varepsilon}(p(t) - p(x))$$

By the property of φ, $\exists\, t$ with $0 < t - x < r$

s.t. $\quad \varphi(t) - \varphi(x) \leq -\frac{1}{2}$

Since $r < \frac{1}{n} \Rightarrow t \in (x, x+\frac{1}{n})$ and s.t.

$$-\frac{1}{2} \geq \frac{3\alpha}{\varepsilon}(t-x) - \frac{3}{\varepsilon}(p(t) - p(x))$$

$$\Rightarrow \quad 1 \leq \frac{12}{\varepsilon}(L+n)r < 1 \quad \text{as before.}$$

Again, it is a contradiction.

Therefore $g \notin A_n$. Hence $g \in B_\varepsilon^\infty(f) \cap [C[a,b] \setminus A_n]$,

which implies $B_\varepsilon^\infty(f) \cap [C[a,b] \setminus A_n] \neq \emptyset$.

This completes the proof of the Theorem. ✗✗

Def: A function $f: [a,b] \to \mathbb{R}$ is said to be <u>nowhere monotonic</u> if $\nexists$ no interval $[c,d] \subset [a,b]$ on which f is monotonic

Cor: The set of <u>continuous nowhere monotonic functions</u> is a residual set in $C[a,b]$, & hence dense in $C[a,b]$.

Pf: If $f \in C[a,b]$ is monotonic on some interval $[c,d]$,

then $LX \equiv \beta$ with $\beta \in (f(c), f(d))$ crosses f

if $f(d) > f(c)$ (or $\beta \in (f(d), f(c))$ if $f(c) > f(d)$)

If $f(c) = f(d)$, then $f \equiv$ const. on $[c,d]$.

Clearly many lines cross f.

Hence f monotonic on some interval

$$\Rightarrow f \in C[a,b] \setminus Z$$

Since $C[a,b] \setminus Z$ is of 1^{st} category,

any subset of $C[a,b] \setminus Z$ is also of 1^{st} category.

$\Rightarrow$ set of cts functions monotonic on some interval

is of 1st category

$\therefore$ Set of cts nowhere monotonic functions is a residual

and hence dense by Baire Category Thm.

(since $C[a,b]$ is complete.) ※

Remark : The Thm can be used to prove Thm 4.13 too.

Another application of Baire Category Theorem

Thm 4.14 Every basis of an infinite dimensional Banach space consists of uncountably many vectors.

Pf: Let V be a Banach space.

Suppose on the contrary that V has a countable basis $\mathcal{B} = \{w_j\}_{j=1}^{\infty}$.

Then $V = \bigcup_{n=1}^{\infty} W_n$

where $W_n = \operatorname{span}\{w_1, \cdots, w_n\}$

Claim 1: W_n has empty interior

Pf: Since V is of infinite dimensional,

$$V \setminus W_n \neq \emptyset, \quad \forall n = 1, 2, \cdots$$

$$\Rightarrow \quad \{v \in V : \|v\| = 1\} \setminus W_n \neq \emptyset, \quad \forall n = 1, 2, \cdots$$

(by scaling & W_n is a vector subspace.)

$\therefore$ one can find $v_0 \in V \setminus W_n$ such that

$$|v_0| = 1.$$

Then $\forall\, w \in W_n$ and $\varepsilon > 0$,

$$w + \frac{\varepsilon}{2} v_0 \in B_\varepsilon(w) \cap (V \setminus W_n)$$

$$\Rightarrow \quad B_\varepsilon(w) \cap (V \setminus W_n) \neq \emptyset$$

$\therefore$ W_n has empty interior.

<u>Claim 2</u> W_n is closed, $\forall\, n = 1, 2, \cdots$

<u>Pf</u>: Let $\{v_j\}_{j=1}^{\infty}$ be a seq in W_n and converges to some $v_0 \in V$.

Note that

$$\begin{array}{ccc} T: W_n & \to & \mathbb{R}^n \\ \Psi & & \Psi \\ \sum_{j=1}^{n} a_j w_j & \mapsto & (a_1, \cdots, a_n) \end{array}$$

is a vector space isomorphism.

And hence the norm in V, $\left|\sum_{j=1}^{n} a_j w_j\right|_V$ gives a norm on $\mathbb{R}^n$

$$\|(a_1, \cdots, a_n)\| = \left|\sum_{j=1}^{n} a_j w_j\right|_V.$$

Since any two norms on $\mathbb{R}^n$ are equivalent,

$\|(a_1, \cdots, a_n)\|$ is equivalent to standard Euclidean norm

$$|(a_1, \cdots, a_n)| = \sqrt{a_1^2 + \cdots + a_n^2}$$

$\Rightarrow \quad \exists\ C_1, C_2 > 0$ s.t.

$$|v|_V \leq C_1 |Tv| \leq C_2 |v|_V \quad , \quad \forall v \in W_n$$

Since $v_\ell \to v_0$ in V, $\{v_\ell\}$ is Cauchy in $(V, |\cdot|_V)$

$\therefore\ \forall \varepsilon > 0, \exists\ l_0 \geq 0$ s.t.

$$|v_\ell - v_k|_V < \varepsilon \quad , \quad \forall \ell, k \geq l_0$$

$$\Rightarrow \quad |Tv_\ell - Tv_k| \leq \frac{C_2}{C_1} |v_\ell - v_k| < \frac{C_2}{C_1} \varepsilon \ , \ \forall \ell, k \geq l_0$$

$\Rightarrow \quad \{Tv_\ell\}$ is Cauchy in $\mathbb{R}^n$ (with standard metric)

By completeness of $\mathbb{R}^n$, $\exists\ a^* = (a_1^*, \cdots, a_n^*) \in \mathbb{R}^n$

s.t. $\quad |Tv_\ell - a^*| \to 0 \quad$ as $\ell \to \infty$.

Let $\quad v^* = T^{-1} a^* = \sum_{j=1}^{n} a_j^* w_j \in W_n$,

we have

$$|v_\ell - v^*|_V \leq C_1 |Tv_\ell - a^*| \to 0 \quad \text{as } \ell \to \infty$$

By uniqueness of limit $v_0 = v^*$ $\Rightarrow$ $v_0 \in W_n$.

$\therefore$ W_n is closed. This proves Claim 2.

By Claims 1 & 2, W_n is nowhere dense

$\therefore$ $V = \bigcup_{n=1}^{\infty} W_n$ is of 1^{st} category.

But V is complete, this is impossible.

(contradicting Baire Category Thm)

Hence any basis of V cannot be countable. #