Applications of Baire Category Theorem (to function spaces)

Thm 4.13 The set of all continuous, nowhere differentiable functions forms a residual set in $C[a,b]$ and hence dense in $C[a,b]$.

To prove the theorem, we need a lemma:

Lemma 4.2: Let $f \in C[a,b]$ be differentiable at x. Then it is Lipschitz continuous at x

(i.e. $|f(y)-f(x)| \le L|y-x|$, $\forall y \in [a,b]$.
It is clear for y near x. The issue is for y not near x)

Pf: By assumption ($\forall \varepsilon > 0$, say $\varepsilon = 1$)

$\exists \delta_0 > 0$ such that

$\forall y \in (x-\delta_0, x+\delta_0) \setminus \{x\}$ (& $y \in [a,b]$)

$$\left| \frac{f(y)-f(x)}{y-x} - f'(x) \right| < 1 .$$

$$\Rightarrow \quad |f(y)-f(x)| \le (1+|f'(x)|)\,|y-x|$$

$$\forall y \in (x-\delta_0, x+\delta_0) \cap [a,b]$$

If $[a,b] \setminus (x-\delta_0, x+\delta_0) = \phi$, we are done.

If not, then for $y \in [a,b] \setminus (x-\delta_0, x+\delta_0)$,

$$|y-x| \geq \delta_0$$

and hence

$$|f(y)-f(x)| \leq |f(y)| + |f(x)|$$

$$\leq 2\|f\|_\infty \leq \frac{2\|f\|_\infty}{\delta_0} |y-x|$$

Let $L = \max\left\{ 1+|f'(x)|, \frac{2\|f\|_\infty}{\delta_0} \right\}$, we have

$$|f(y)-f(x)| \leq L\, |y-x|, \quad \forall\, y \in [a,b]. \quad \#$$

<u>Pf of Thm 4.13</u>

We only need to show the case that $[a,b] = [0,1]$.

$\forall\, L > 0$, define

$$S_L = \left\{ f \in C[0,1] : \begin{array}{l} f \text{ is Lip. cts at some } x \in [0,1] \\ \text{with Lip. Const.} \leq L \end{array} \right\}$$

<u>Claim 1</u>: S_L is closed.

Pf: Let $\{f_n\}$ be a seq. in S_L which converges to some $f \in C[0,1]$ in d_∞ metric.

By definition of S_L, $\forall n \geq 1$

$\exists\, x_n \in [0,1]$ such that

f_n is Lip. cts at x_n with Lip const $\leq L$

i.e. $|f_n(y) - f_n(x_n)| \leq L|y - x_n|$, $\forall y \in [0,1]$.

We may assume that $x_n \to x^*$ for some $x^* \in [0,1]$

by passing to a subseq.

(The corresponding subseq. f_n is still convergent

& $f_n \to f$ in d_∞)

Then

$$|f(y) - f(x^*)| \leq |f(y) - f_n(y)| + |f_n(y) - f(x^*)|$$

$$\leq \|f - f_n\|_\infty + |f_n(y) - f_n(x_n)| + |f_n(x_n) - f(x^*)|$$

$$\leq \|f - f_n\|_\infty + L|y - x_n| + |f_n(x_n) - f_n(x^*)| + |f_n(x^*) - f(x^*)|$$

$$\leq 2\|f - f_n\|_\infty + L|y - x_n| + L|x_n - x^*|$$

$$\leq 2\|f - f_n\|_\infty + L|y - x^*| + L|x^* - x_n| + L|x_n - x^*|$$

$$= L|y - x^*| + 2(\|f - f_n\|_\infty + L|x_n - x^*|)$$

Letting $n \to +\infty$, we have

$$|f(y) - f(x^*)| \leq L|y - x^*| \quad , \quad \forall y \in [0,1]$$

$$\Rightarrow f \in S_L.$$

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(To be cont'd)