(c) (cont'd)

Assume (X, d) has no isolated point,

<u>Claim</u>: $\forall x \in X$, $\{x\}$ is nowhere dense in X.

<u>Pf</u>: Suppose not, then $\overline{\{x\}}$ $(=\{x\})$ contains some open ball $B_r(y)$,

i.e. $B_r(y) \subset \overline{\{x\}} = \{x\}$

This implies, $\begin{cases} y = x \quad \& \\ B_r(x) \subset \{x\} \subset B_r(x). \end{cases}$

$\Rightarrow$ $\{x\} = B_r(x)$ is open

$\Rightarrow$ x is isolated which is a contradiction,

Then by (b) & the claim, any finite set is nowhere dense.

※

<u>eg</u>: $(\mathbb{R}, d(x,y) = |x-y|)$ has no isolated point

$\Rightarrow$ any $\{x_1, \cdots, x_n\}$ is nowhere dense.

But for countable subsets, we have no such conclusion:

- $\mathbb{N} = \{1, 2, 3, \cdots\}$ countable and nowhere dense (Ex!)
- $\mathbb{Q}$ countable, but <u>not</u> nowhere dense. (in fact, $\mathbb{Q}$ is dense)

Examples in infinite dimensional normed spaces

eg: Let $M[a,b]$ = space of bounded functions on $[a,b]$. (Not necessary continuous)

Then $\|f\|_\infty = \sup_{[a,b]} |f(x)|$ is well-defined

and is a norm on $M[a,b]$. (check!)

Clearly $(C[a,b], d_\infty)$ is a metric (also vector) subspace of

$(M[a,b], d_\infty)$

Claim: $C[a,b]$ is nowhere dense in $M[a,b]$ (wrt d_∞ metric).

Pf: (1) Clearly, $C[a,b]$ is closed in $M[a,b]$

(uniform limit of cts. functions is cts.)

Hence $C[a,b]$ is nowhere dense in $M[a,b]$

$\Leftrightarrow$ $M[a,b] \setminus \overline{C[a,b]} = M[a,b] \setminus C[a,b]$ is dense.

$\therefore$ We only need to show that:

(2) $\forall$ $B_\varepsilon^\infty(f) \subset M[a,b]$, $B_\varepsilon^\infty(f) \cap \left(M[a,b] \setminus C[a,b]\right) \neq \phi$.

(i) If $f \in M[a,b] \setminus C[a,b]$, we are done.

(ii) If $f \in C[a,b]$,

define $g(x) = \begin{cases} f(x) + \frac{\varepsilon}{2}, & x \in [a,b] \cap \mathbb{Q} \\ f(x) - \frac{\varepsilon}{2}, & x \in [a,b] \setminus \mathbb{Q}. \end{cases}$

Then $g(x) - f(x) = \pm \frac{\varepsilon}{2}$

$\Rightarrow \quad \|g-f\|_\infty = \frac{\varepsilon}{2} \quad \Rightarrow \quad g \in B_\varepsilon^\infty(f)$

If $g \in C[a,b]$, then

$$g - f = \begin{cases} \varepsilon/2, & [a,b] \cap \mathbb{Q} \\ -\varepsilon/2, & [a,b] \setminus \mathbb{Q} \end{cases} \quad \text{is continuous,}$$

which is impossible.

Hence $g \in M[a,b] \setminus C[a,b]$

$$\Rightarrow \quad B_\varepsilon^\infty(f) \cap (M[a,b] \setminus C[a,b]) \neq \emptyset$$

$\divideontimes$

eg: Let l_∞ = space of bounded sequences with d_∞ metric

$$d_\infty(x,y) = \sup_n |x_n - y_n| \quad \text{for } x = \{x_n\},\ y = \{y_n\}$$

Let $\mathcal{C}$ = subset of convergent sequences.

Then $\mathcal{C}$ is nowhere dense in (l_∞, d_∞).

Pf: We only need to show (1) & (2) in the following

(1) $\mathcal{C}$ is closed in l_∞.

Pf: (We'll show that $l_\infty \setminus \mathcal{C}$ is open)

Let $x = \{x_n\} \in l_\infty \setminus \mathcal{C}$

Then x_n diverges and

$$(+\infty >)\ L = \limsup_n x_n > \liminf_n x_n = l \ (> -\infty)$$

Take $\varepsilon = \frac{L-l}{3} > 0$

then $\forall\, y = \{y_n\} \in B^\infty_\varepsilon(x)$, we have

$$x_n - \varepsilon < y_n < x_n + \varepsilon \quad , \quad \forall n$$

$$\begin{cases} \limsup x_n - \varepsilon \le \limsup y_n \\ \liminf y_n \le \liminf x_n + \varepsilon \end{cases}$$

$$\Rightarrow \quad \limsup y_n \ge L - \varepsilon = \frac{2L+l}{3} > \frac{L+2l}{3} \quad (\text{since } L > l)$$

$$= l + \varepsilon \ge \liminf y_n$$

$\Rightarrow$ $y = \{y_n\}$ is divergent.

Hence $B^\infty_\varepsilon(x) \subset l_\infty \setminus \mathcal{C} \Rightarrow l_\infty \setminus \mathcal{C}$ is open

& this proves (1)

(2) $\ell_\infty \setminus \mathcal{C}$ $\left(= \ell_\infty \setminus \overline{\mathcal{C}} \text{ by (1)}\right)$ is dense

Pf: Let $B^\infty_\varepsilon(x)$ be a ball in ℓ_∞,

we need to show that $B^\infty_\varepsilon(x) \cap (\ell_\infty \setminus \mathcal{C}) \neq \phi$

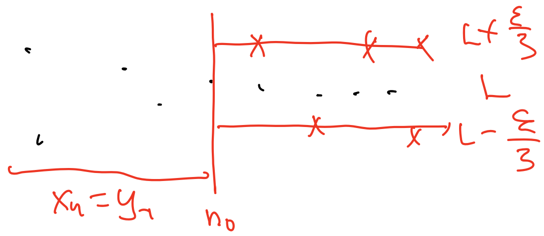
If $x \in \ell_\infty \setminus \mathcal{C}$, we are done.

If $x \in \mathcal{C}$, then $x = \{x_n\}$ is convergent

Let $L = \lim\limits_n x_n$

Then $\exists\, n_0 > 0$ s.t. $|x_n - L| < \frac{\varepsilon}{3}$, $\forall n \geq n_0$.

Define $y = \{y_n\} \in \ell_\infty$ by



$$y_n = \begin{cases} x_n, & \text{if } n < n_0 \\ L + \frac{\varepsilon}{3}, & \text{if } n \geq n_0 \text{ \& } n \text{ odd} \\ L - \frac{\varepsilon}{3}, & \text{if } n \geq n_0 \text{ \& } n \text{ even}. \end{cases}$$

Then $|x_n - y_n| = 0$ if $n < n_0$ and

$$|x_n - y_n| \leq |x_n - L| + |L - y_n|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} < \varepsilon, \quad \forall n \geq n_0$$

$$\Rightarrow \quad d_\infty(x, y) \leq \frac{2\varepsilon}{3} < \varepsilon \quad \Rightarrow \quad y \in B^\infty_\varepsilon(x)$$

However $\limsup y_n = L + \frac{\varepsilon}{3} > L - \frac{\varepsilon}{3} = \liminf y_n$.

$\therefore$ $y \in \ell_\infty \setminus \mathcal{C}$. $\Rightarrow$ $B^\infty_\varepsilon(x) \cap (\ell_\infty \setminus \mathcal{C}) \neq \phi$ #

Def: • A set in a metric space is called <u>of first catogroy</u> (or <u>meager</u>) if it can be expressed as a <u>countable union of nowhere dence</u> sets.

• A set is <u>of second category</u> if it is <u>not</u> of first categroy.

• A set is called <u>residual</u> if its <u>complement</u> is of first category.

Prop 4.8 Let (X, d) be a metric space.

(a) Every subset of a set of 1^{st} category is of 1^{st} category.

(b) The union of <u>countable</u> many sets of 1^{st} category is of 1^{st} category

(c) If (X, d) has no isolated point, then every <u>countable</u> subset of X is of 1^{st} category.

Pf: (a) Let $E \subset X$ be a set of 1^{st} category.

Then $E = \bigcup_{n=1}^{\infty} E_n$ for some nowhere dense sets E_n, $n=1,2,\cdots$

Let $F \subset E$, then by Prop 4.7 (a)

$F \cap E_n$ is nowhere dense, $\forall n$ $(F \cap E_n \subset E_n)$

Hence $F = F \cap E = \bigcup_{n=1}^{\infty} (F \cap E_n)$ is of 1^{st} category.

(b) Let $E_n = \bigcup_{k=1}^{\infty} E_{n,k}$, $E_{n,k}$ = nowhere dense.

$$\Rightarrow \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=1}^{\infty} E_{n,k} \right) = \bigcup_{(n,k) \in \mathbb{N} \times \mathbb{N}} E_{n,k}$$

is of 1^{st} category. (since $\mathbb{N} \times \mathbb{N}$ is countable)

(c) If $E = \{x_i\}_{i=1}^{\infty} \subset X$, then Prop 4.7 (c)

$\Rightarrow$ $\{x_i\}$ is nowhere dense $\forall i$

$\Rightarrow$ $E = \bigcup_{i=1}^{\infty} \{x_i\}$ is of 1^{st} category (by part (b)) #

<u>Prop 4.8'</u> Let (X, d) be a metric space.

(a) Every subset containing a residual set is residual.

(b) The intersection of <u>countable</u> many residual sets is a residual set.

(c) If (X, d) has no isolated point, then complement of a countable set is a residual set.

(Pf: By taking complement in Prop 4.8)

eg 4.5: $\mathbb{R}$ has no isolated point (in standard metric)

$\Rightarrow$ $\{q_j\}$ is nowhere dense $\forall$ rational number

$\Rightarrow$ $\mathbb{Q}$ is of 1^{st} category

Hence $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ the set of irrational numbers is a residual set in $\mathbb{R}$.

Thm 4.9 (Baire Category Theorem)

In a complete metric space, any set of 1^{st} category has empty interior.

Pf: Let the complete metric space be (X, d).

And let $E = \bigcup_{n=1}^{\infty} E_n \subset X$ be of 1^{st} category

where $\overline{E_n}$ is nowhere dense in X, $\forall n$

Consider any open metric ball $B_{r_0}(x_0)$ of X.

Since $\overline{E_1}$ has empty interior (by defn. of nowhere denseness),

$$(X \setminus \overline{E_1}) \cap B_{r_0}(x_0) \neq \emptyset$$

let $x_1 \in (X \setminus \overline{E_1}) \cap B_{r_0}(x_0)$.

Since both $X \setminus \overline{E_1}$ & $B_{r_0}(x_0)$ are open,

$\exists\, r_1 > 0$ s.t. $\quad \overline{B_{r_1}(x_1)} \subset (X \setminus \overline{E_1}) \cap B_{r_0}(x_0)$

and $\quad r_1 \leq \frac{r_0}{2} \quad$ (as we can always choose a smaller ball, if $B_r(x_1) = B_{r_1}(x_1)$, $\forall\, r < r_1$ then $B_{r_1}(x_1) = \{x_1\} \Rightarrow \overline{B_{r_1}(x_1)} = B_{r_1}(x_1)$)

$\Rightarrow \quad \overline{B_{r_1}(x_1)} \cap \overline{E_1} = \emptyset$

Now E_2 is nowhere dense, $\overline{E_2}$ has empty interior.

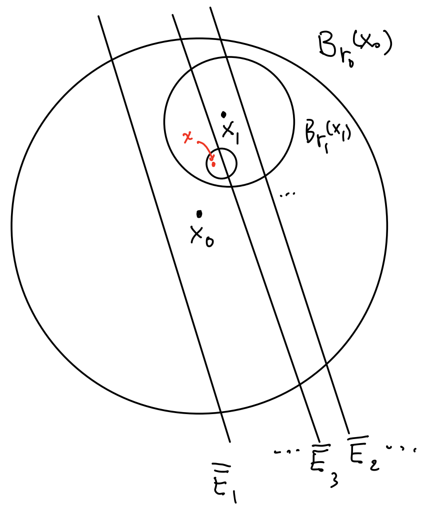
$\Rightarrow \quad (X \setminus \overline{E_2}) \cap B_{r_1}(x_1) \neq \emptyset$.

Similarly to the above, $\exists\, x_2 \in (X \setminus \overline{E_2}) \cap B_{r_1}(x_1)$

and $r_2 > 0$ with $\quad r_2 \leq \frac{r_1}{2} \quad$ such that

$$\overline{B_{r_2}(x_2)} \subset (X \setminus \overline{E_2}) \cap B_{r_1}(x_1) \qquad \left(\begin{array}{l} \subset (X \setminus \overline{E_2}) \\ \subset B_{r_1}(x_1) \end{array} \right)$$

Note that $\quad \overline{B_{r_2}(x_2)} \subset B_{r_1}(x_1) \subset (X \setminus \overline{E_1}) \cap B_{r_0}(x_0) \subset X \setminus \overline{E_1}$.



Repeating the process, we obtain $\{x_n\}_{n=1}^{\infty} \subset X$

and $\{r_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$ such that

(a) $\overline{B_{r_{n+1}}(x_{n+1})} \subset B_{r_n}(x_n)$

(b) $r_{n+1} \leq \frac{r_n}{2}$

(c) $\overline{B_{r_n}(x_n)} \subset X \setminus \overline{E_j}$, $\forall j = 1, \cdots, n$

$\left(\overline{B_{r_n}(x_n)} \cap \overline{E_j} = \phi, \ \forall j = 1, \cdots, n \right)$

By (a) & (b), $\{x_n\}$ is a Cauchy seq. (Ex!)

Hence completeness of $X \Rightarrow \exists\, x \in X$ s.t. $x_n \to x$.

By (a) again, $x_{n+m} \in \overline{B_{r_n}(x_n)}$, $\forall m = 1, 2, 3, \cdots$

$\Rightarrow \quad x \in \overline{B_{r_n}(x_n)}$

By (a) & (c) $\quad x \in X \setminus \overline{E_n}$ and $B_{r_0}(x_0)$

Since n is arbitrary, $x \in \bigcap_{n=1}^{\infty} (X \setminus \overline{E_n}) = X \setminus \left(\bigcup_{n=1}^{\infty} \overline{E_n} \right)$

$\Rightarrow \quad x \in \left(X \setminus \left(\bigcup_{n=1}^{\infty} \overline{E_n} \right) \right) \cap B_{r_0}(x_0)$

$\Rightarrow \quad \left(X \setminus \left(\bigcup_{n=1}^{\infty} \overline{E_n} \right) \right) \cap B_{r_0}(x_0) \neq \phi$

$$\Rightarrow \left(X \setminus \left(\bigcup_{n=1}^{\infty} E_n\right)\right) \cap B_{r_0}(x_0) \supset \left(X \setminus \left(\bigcup_{n=1}^{\infty} \overline{E_n}\right)\right) \cap B_{r_0}(x_0)$$

$$\neq \phi$$

Since $B_{r_0}(x_0)$ is arbitrary, $E = \bigcup_{n=1}^{\infty} E_n$ has empty interior.

$\#$

> Note : This implies, if X complete, residue set is dense
>
> (E empty interior $\Rightarrow$ $X \setminus E$ dense)

Recall that E is closed nowhere dense set

$\Leftrightarrow$ $X \setminus E$ ($= X \setminus \overline{E}$) is an open dense set.

Hence Thm 4.9 can be rephased as

> Thm 4.9′ (Baire Category Theorem)
>
> In a complete metric space, countable intersection of open dense sets is dense.

i.e. If (X, d) is complete and $G_n \subset X$ is a sequence of open dense sets in X, then $\bigcap_{n=1}^{\infty} G_n$ is dense.

(Pf : Ex!)

Cor 4.10: Let (X, d) be complete.

Suppose that $X = \bigcup_{n=1}^{\infty} E_n$ with E_n are closed subsets.

Then at least one of these E_n's has non-empty interior.

Pf: Suppose not, then all E_n has empty interior.

$\Rightarrow$ E_n is nowhere dense, $\forall n$. (Since E_n are closed)

Hence $X = \bigcup_{n=1}^{\infty} E_n$ is of 1^{st} category.

Baire Category Thm $\Rightarrow$ X has empty interior which is a contradiction since $X^{\circ} = X$. ※

Remark: This corollary implies that it is impossible to decompose a complete metric space into a countable union of nowhere dense sets.

(i.e. complete metric space itself is of 2^{nd} category.)

$\boxed{\text{Cor 4.11 A set of 1st category in a } \underline{\text{complete}} \text{ metric space } \underline{\text{cannot}} \text{ be a residual set, and vice versa.}}$

($\Rightarrow$ residual sets of a complete metric space is of 2nd category)

Pf: Let E be a set of 1st category,

then $E = \bigcup_{n=1}^{\infty} E_n$ with E_n nowhere dense.

If E is also a residual set,

then $X \setminus E$ is also of 1st category,

hence $X \setminus E = \bigcup_{n=1}^{\infty} E_n'$ with E_n' nowhere dense.

$$\Rightarrow X = E \cup (X \setminus E) = \left(\bigcup_{n=1}^{\infty} E_n\right) \cup \left(\bigcup_{n=1}^{\infty} E_n'\right)$$

Taking closure of E_n & E_n',

$$X \subset \left(\bigcup_{n=1}^{\infty} \overline{E_n}\right) \cup \left(\bigcup_{n=1}^{\infty} \overline{E_n'}\right) \quad (\subset X)$$

$$\Rightarrow X = \left(\bigcup_{n=1}^{\infty} \overline{E_n}\right) \cup \left(\bigcup_{n=1}^{\infty} \overline{E_n'}\right)$$

i.e. X is a countable union of close subsets with empty interiors. This contradicts Cor 4.10.

The other way is similar. $\#$

eg: $\mathbb{R}$ is complete, $\mathbb{Q}$ of 1^{st} category

$\Rightarrow \mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ is of 2^{nd} category.