§4.2 Baire Category Theorem

Def let (X, d) be a metric space. A set E in X is dense if $\forall\, x \in X$ and $\varepsilon > 0$,

$$B_\varepsilon(x) \cap E \neq \emptyset$$

Notes: (i) Easy to see that E is dense $\Leftrightarrow \overline{E} = X$.

(ii) X is dense (in (X, d))

eg: If $(X$, discrete metric$)$, then for $0 < \varepsilon < 1$ and $x \in X$,

$B_\varepsilon(x) = \{x\}$. Therefore E is dense in $X \Rightarrow E = X$

(i.e. X is the only dense set in $(X$, discrete$)$)

eg1: In $(\mathbb{R}$, standard metric$)$, $\mathbb{Q}$ and $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ are dense.

eg2: Weierstrass approximation theorem implies the set of all polynomials $\mathcal{P}$ forms a dense set in $(C[0,1], d_\infty)$.

Def: Let (X, d) be a metric space. A subset $E \subset X$ is called

<u>nowhere dense</u> if

its <u>closure</u> does <u>not</u> contain any metric ball.

(i.e. $\overline{E}$ has empty interior $(\overline{E})^\circ = \phi$)

eg • $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ nowhere dense in $\mathbb{R}$.

• $\mathbb{Q}$ has empty interior, but $\overline{\mathbb{Q}} = \mathbb{R}$ has nonempty interior,

so $\mathbb{Q}$ is <u>not</u> nowhere dense.

<u>Note:</u> E is nowhere dense $\iff$ $X \setminus \overline{E}$ is dense in X

<u>Pf</u>: E is nowhere dense

$\iff$ $\forall x \in X$ & $r > 0$, $B_r(x) \not\subset \overline{E}$ (since $\overline{E}$ contains no ball)

$\iff$ $\forall x \in X$ & $r > 0$, $B_r(x) \cap (X \setminus \overline{E}) \neq \emptyset$

$\iff$ $X \setminus \overline{E}$ is dense.

Def: Let (X, d) be a metric space. A point $x \in X$ is called

an <u>isolated point</u> if $\{x\}$ is open in X.

Notes: • As $\{x\}$ is always closed in a metric space,

$\{x\}$ is both open and closed in X

$\Leftrightarrow$

x is an isolated point.

• x isolated $\Rightarrow$ $\{x\}$ is not nowhere dense.

egs: • $\mathbb{R}$ has no isolated points (since $\{x\}$ is not open in $\mathbb{R}$, $\forall x \in \mathbb{R}$)

• All points in $\mathbb{Z}$ (subspace of $\mathbb{R}$) are isolated in $\mathbb{Z}$ (not $\mathbb{R}$).

since: $\forall n \in \mathbb{Z}$, $\{n\} = B_{\frac{1}{2}}(n)$ metric ball in $\mathbb{Z}$

(But $(\mathbb{Z}$, subspace metric$) \neq (\mathbb{Z}$, discrete metric$)$)

unbounded — bounded ≤ 1

Prop 4.7 Let (X,d) be a metric space.

(a) E is nowhere dense in X $\Rightarrow$

(i) $\overline{E}$ is nowhere dense in X ;

(ii) if $E' \subset E \Rightarrow E'$ is nowhere dense in X

(b) The union of finite many nowhere dense sets (in X) is nowhere dense (in X)

(c) If (X,d) has no isolated point, then every finite set is nowhere dense.

Pf: (a) Trivial

(b) Let E_1, E_2 be nowhere dense sets

Then $G_1 = X \setminus \overline{E_1}$ and $G_2 = X \setminus \overline{E_2}$ are open dense set.

Clearly $G_1 \cap G_2$ is open.

claim: $G_1 \cap G_2$ is dense in X.

pf: $\forall x \in X$ & $r > 0$,

G_1 dense $\Rightarrow B_r(x) \cap G_1 \neq \emptyset$

$\Rightarrow \exists\, x_1 \in B_r(x) \cap G_1$.

Since $B_r(x) \cap G_1$ is open, $\exists\ \rho > 0$ such that

$$B_\rho(x_1) \subset B_r(x) \cap G_1 .$$

Now G_2 dense $\Rightarrow$ $B_\rho(x_1) \cap G_2 \neq \phi$

$$\Rightarrow \quad B_r(x) \cap (G_1 \cap G_2) \supset B_\rho(x_1) \cap G_2 \neq \phi$$

This proves the claim.

Hence $\quad X \setminus (G_1 \cap G_2) = (X \setminus G_1) \cup (X \setminus G_2)$

$$= \overline{E_1} \cup \overline{E_2} \quad \text{is nowhere dense.}$$

By (a)(ii), $E_1 \cup E_2 \subset \overline{E_1} \cup \overline{E_2} \Rightarrow$

$E_1 \cup E_2$ is also nowhere dense.

Then, induction $\Rightarrow$

$\bigcup_{i=1}^{k} E_i$ is nowhere dense provided $E_1, \cdots, E_k$ are nowhere dense.

(c) (to be cont'd)