Remarks

(1) Ascoli's Theorem remains valid for bounded and equicontinuous subsets of $C(G)$.

(i.e. No need to take closure of G.)

It is because

"equicontinuous" $\Rightarrow$ "uniform continuous on G",

and then, can be extended to uniform continuous on $\overline{G}$.

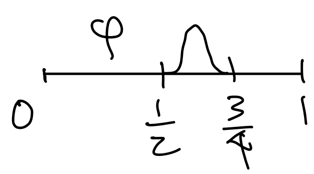
(and equicontinuous of $C(\overline{G})$)

(Details omitted.)

(2) However, <u>boundedness</u> of the domain $\overline{G}$ cannot be removed:

<u>Eg 4.3</u> Let $\overline{G} = [0, \infty) \subset \mathbb{R}$.

Take a $\varphi \in C^1[0,1]$ such that



- $\varphi \not\equiv 0$ and
- $\varphi(x) = 0$ on $[0,1] \setminus [\frac{1}{2}, \frac{3}{4}]$

and define

$$f_n(x) = \begin{cases} \varphi(x-n), & \text{if } x \in [n, n+1] \\ 0, & \text{otherwise.} \end{cases}$$

Then one can easily check that

$$f_n \in C(\bar{G}) \qquad (\text{in fact } f_n \in C^1(\bar{G}))$$

and $\|f_n\|_{\infty,\bar{G}} = \|\varphi\|_{\infty,[0,1]} \,(>0)$ (a fixed constant)

$\therefore$ $\mathcal{E} = \{f_n\}$ is bounded subset in $C(\bar{G})$.

By Chain rule,

$$\left\|\frac{df_n}{dx}\right\|_{\infty,\bar{G}} = \left\|\frac{d\varphi}{dx}\right\|_{\infty,[0,1]} (>0.) \quad \text{indep. of } n.$$

Hence Prop 4.1 implies that

$\mathcal{E} = \{f_n\}$ is also equicontinuous.

On the other hand, suppose $\exists$ subsequence $\{f_{n_j}\}$ of $\{f_n\}$

converges to some $f \in C(\bar{G})$ in d_∞, i.e. $f_{n_j} \to f$ uniformly on $\overline{G}$,

which implies pointwise convergence

$$f_{n_j}(x) \to f(x), \quad \forall x \in \overline{G}.$$

Since for any fixed $x \in \overline{G}$

$$f_n(x) = 0, \quad \forall n \geq x,$$

we must have

$$\lim_{j \to +\infty} f_{n_j}(x) = 0.$$

$$\therefore \quad f(x) = 0, \quad \forall x \in \overline{G}.$$

This is a contradiction as this implies

$$0 < \|\varphi\|_{\infty,[0,1]} = \|f_{n_j}\|_{\infty,\overline{G}} = \|f_{n_j} - f\|_{\infty,\overline{G}} \to 0.$$

$\therefore$ $\mathcal{E}$ is not precompact.

Hence Ascoli's Theorem doesn't hold. #

Converse to Ascoli's Theorem:

Thm 4.4 (Arzela's Theorem)

Suppose that G is a bounded nonempty open set in $\mathbb{R}^m$.

Then every precompact set in $C(\bar{G})$ must be bounded

and equicontinuous.

Pf: Let $\mathcal{E} \subset C(\bar{G})$ be precompact.

If $\mathcal{E}$ is unbounded, then $\exists f_n \in \mathcal{E} \subset C(\bar{G})$

such that $\lim_{n \to +\infty} \|f_n\|_\infty = \infty$.

Then this subset $\{f_n\}$ of $\mathcal{E}$ cannot contain any convergent subsequence. This contradicts the precompactness.

Hence $\mathcal{E}$ must be bounded.

Now suppose on the contrary that $\mathcal{E}$ is precompact, bounded but not equicontinuous.

Then $\exists\, \varepsilon_0 > 0$ such that $\forall\, \delta > 0$

$\exists\, x, y \in \overline{G}$ and $f \in \mathcal{E}$ satisfying

$$|f(x) - f(y)| \geq \varepsilon_0 \ \& \ d(x,y) < \delta.$$

In particular, by choosing $\delta = \frac{1}{n} > 0$, for $n = 1, 2, \cdots$

$\exists\, x_n, y_n \in \overline{G}$ and $f_n \in \mathcal{E}$ satisfying

$$|f_n(x_n) - f_n(y_n)| \geq \varepsilon_0 \ \& \ d(x_n, y_n) < \frac{1}{n}.$$

By precompactness, $\exists$ convergent subseq. $\{f_{n_k}\}$ of $\{f_n\}$.

Suppose $f \in C(\overline{G})$ is the limit,

i.e. $\quad d_\infty(f_{n_k}, f) \to 0$, as $k \to +\infty$.

(i.e. f_{n_k} converges uniformly to f on $\overline{G}$)

Since $\overline{G}$ is closed and bounded, the corresponding sequences

of points $\{x_{n_k}\}$ ($\{y_{n_k}\}$) contains convergent subsequence.

Denotes the subseq. by $\{x_k\}$ and assume $x_k \to z \in \overline{G}$.

And also denote the corresponding subseq. of $\{y_{n_k}\}$ by $\{y_k\}$, and the corresponding subseq. of $\{f_{n_k}\}$ by $\{g_k\}$.

Then
$$\begin{cases} g_k \to f \text{ in } (C(\bar{G}), d_\infty) \\ x_k \to z \text{ in } \bar{G} \end{cases}$$

Since $d(x_n, y_n) < \frac{1}{n}$, we have

$$d(x_k, y_k) \to 0 \text{ as } k \to \infty$$

and hence $y_k \to z \in \bar{G}$ too.

Therefore, $\forall \varepsilon > 0$, $\exists k_0 \geq 0$ s.t.

$$\|g_k - f\|_\infty < \varepsilon, \ \forall k \geq k_0.$$

and $\exists k_1 \geq 0$ s.t.

$$\begin{cases} |f(x_k) - f(z)| < \varepsilon \\ |f(y_k) - f(z)| < \varepsilon \end{cases} \quad \forall k \geq k_1$$

Hence for $k \geq \max\{k_0, k_1\}$,

$$|g_k(x_k) - g_k(y_k)| \leq |g_k(x_k) - f(x_k)| + |f(x_k) - f(y_k)| + |f(y_k) - g_k(y_k)|$$

$$< 2\varepsilon + |f(x_k) - f(y_k)|$$

$$\leq 2\varepsilon + |f(x_k) - f(z)| + |f(z) - f(y_k)|$$

$$< 4\varepsilon$$

We've show that $\forall\, \varepsilon > 0$, $\exists\, n_0 = n_{\max\{k_0, k_1\}} \geq 0$ such that

$$|f_{n_k}(x_{n_k}) - f_{n_k}(y_{n_k})| < 4\varepsilon, \quad \forall\, n_k \geq n_0$$

Taking $\varepsilon = \frac{\varepsilon_0}{4}$, we have a contradiction.

$\therefore$ $\mathcal{E}$ is equicontinuous. #

Application to Ordinary Differential Equations

Consider

$$(\text{IVP}) \begin{cases} \dfrac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

with f continuous (not necessary Lipschitz) on

$$R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b].$$

Of course, we cannot expect uniqueness result, but short time existence can be proved.

Idea of proof:

(1) Weierstrass Approximation Theorem (on $\mathbb{R}^2$)

$\Rightarrow \exists \{p_n\}$ sequence of polynomials s.t.

$$d_\infty(p_n, f) \to 0 \qquad (\text{in } C(R))$$

(2) Note that $\forall$ p_n satisfies Lipschitz condition (uniform in t).

By Picard-Lindelöf Theorem,

$$\exists\ a_n' > 0 \text{ with } a_n' < \min\{a, \frac{b}{M_n}, \frac{1}{L_n}\},$$

where $M_n = \|f_n\|_{\infty, R}$

L_n = Lipschitz constant of f_n on R,

s.t. $\exists$ unique solution $x_n \in C^1[t_0 - a_n', t_0 + a_n']$ to the approximated (IVP)

$$\begin{cases} \dfrac{dx_n}{dt} = f_n(t, x_n) & \forall t \in [t_0 - a_n', t_0 + a_n'] \\ x_n(t_0) = x_0 \end{cases}$$

(3) Then try to apply Ascoli's Theorem to $\{x_n\}$ and find a convergent subsequence $x_{n_k} \to x$ for some function $x(t)$.

And hope that x is the required solution.

<u>Issue:</u> Since f is not assumed to satisfy the Lipschitz condition, one cannot expect $\{L_n\}$ is bounded

(In fact, it is unbounded. Otherwise f satisfies Lip condition)

Then $\min\{a, \frac{b}{M_n}, \frac{1}{L_n}\} \to 0 \Rightarrow a_n' \to 0$.

We will not have an "interval" for the existence of the solution.

(On the other hand, as $p_n \rightrightarrows f$ in $(C(\mathbb{R}), d_\infty)$, we have

$M_n \leq M$ for some $M > 0$.)

Therefore, to implement our plan, we need to improve the Picard-Lindelöf Theorem to

Prop 4.5 Under the setting of Picard-Lindelöf Theorem,

$\exists$ unique solution $x(t)$ on the interval $[t_0 - a', t_0 + a']$

with $x(t) \in [x_0 - b, x_0 + b]$, where a' is any number satisfying

$$0 < a' < a^* = \min\{a, \frac{b}{M}\}.$$

Clearly, this implies $\exists$ unique solution on the open interval

$(t_0 - a^*, t_0 + a^*)$.

Pf: Omitted

(Instead, we'll see another proof which doesn't used this Picard-Lindelöf Theorem.)

Thm 4.6 (Cauchy-Peano Theorem)

Consider

$$(\text{IVP}) \begin{cases} \dfrac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

where f is continuous on $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$.

There exists $a' \in (0, a)$ and a C^1-function

$$x : [t_0 - a', t_0 + a'] \to [x_0 - b, x_0 + b]$$

solving the (IVP).

Pf: As in the "Idea of Proof",

$\exists$ sequence of polynomials $\{p_n\}$ s.t.

$$p_n \to f \text{ in } (C(R), d_\infty).$$

This implies

$$M_n = \|p_n\|_{\infty, R} \to M, \quad \text{where } M = \|f\|_{\infty, R},$$

and p_n satisfies the Lipschitz condition.

(we don't need to worry about the Lip. constants by Prop 4.5)

By Prop 4.5, $\exists$ unique solution x_n defined on

$$I_n = (t_0 - a_n, t + a_n),$$

$$\text{where } a_n = \min\{a, \frac{b}{M_n}\},$$

for the (IVP)
$$\begin{cases} \frac{dx_n}{dt} = p_n(t, x_n), & t \in I_n. \\ x_n(t_0) = x_0 \end{cases}$$

with
$$x_n(t) \in [x_0 - b, x_0 + b].$$

As $a_n = \min\{a, \frac{b}{M_n}\} \to \min\{a, \frac{b}{M}\} = a^*$, we have

for any fixed $a' < a^*$ $(a' > 0)$, $\exists\, n_0 > 0$ such that

for $n \geq n_0$,

$$[t_0 - a', t_0 + a'] \subset I_n = (t_0 - a_n, t_0 + a_n).$$

Hence $\forall n \geq n_0$, x_n is defined on $[t_0 - a', t_0 + a']$.

Claim 1 : $\{x_n\} \subset C[t_0 - a', t_0 + a']$ is equicontinuous.

In fact, (IVP) $\Rightarrow$ $\left|\frac{dx_n}{dt}\right| = |p_n(t, x_n)| \leq M_n \quad \forall t$

Since $M_n \to M$, $\left\| \frac{dx_n}{dt} \right\|_\infty$ is uniformly bounded.

By Prop 4.1, $\{x_n\}$ is equicontinuous.

Claim 2: $\{x_n\}$ is bounded in $C[t_0 - a', t_0 + a']$

In fact, (IVP) $\Rightarrow$

$$x_n(t) = x_0 + \int_{t_0}^{t} f_n(s, x_n(s))\, ds, \quad \forall t \in [t_0 - a', t_0 + a']$$

$$\therefore \quad |x_n(t)| \leq |x_0| + a' \sup_s |f_n(s, x_n(s))| \leq |x_0| + a' M_n$$

$\Rightarrow$ $\|x_n\|_{\infty, [t_0 - a', t_0 + a']}$ is uniformly bounded.

$\therefore$ $\{x_n\}$ is a bounded set in $C[t_0 - a', t_0 + a']$.

Then Claims 1 & 2 allow us to apply Ascoli's Theorem to conclude that

$\exists$ a subsequence x_{n_j} in $C[t_0 - a', t_0 + a']$ converges

uniformly to a cts. function x on $[t_0 - a', t_0 + a']$.

Claim 3: x solves (IVP) $\begin{cases} \frac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases}$.

Proof of Claim 3 : We only need to show that

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

Note that x_{n_j} satisfies

$$x_{n_j}(t) = x_0 + \int_{t_0}^{t} f_{n_j}(s, x_{n_j}(s))\, ds.$$

Clearly $x_{n_j}(t) \to x(t)$ as $j \to +\infty$.

We only need to show that

$$\lim_{j\to\infty} \int_{t_0}^{t} f_{n_j}(s, x_{n_j}(s))\, ds = \int_{t_0}^{t} f(s, x(s))\, ds.$$

Since $f \in C(R)$ & R is closed & bounded in $\mathbb{R}^2$,

f is uniformly continuous on R.

Therefore, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$\forall (s_1, x_1), (s_2, x_2) \in R$ with $|s_1 - s_2| < \delta$ and $|x_1 - x_2| < \delta$,

we have

$$|f(s_2, x_2) - f(s_1, x_1)| < \varepsilon.$$

On the other hand, $\|p_n - f\|_{\infty, R} \to 0$

$\Rightarrow \exists\, n_0 > 0$ s.t. $\quad |p_n(s,x) - f(s,x)| < \varepsilon, \ \forall (s,x) \in R.$

Therefore, for j sufficiently large such that

$$n_j \geq n_0 \quad \& \quad \|x_{n_j} - x\|_\infty < \delta,$$

we have

$$\left| \int_{t_0}^{t} p_{n_j}(s, x_{n_j}(s))\,ds - \int_{t_0}^{t} f(s, x(s))\,ds \right|$$

$$\leq \left| \int_{t_0}^{t} p_{n_j}(s, x_{n_j}(s))\,ds - \int_{t_0}^{t} f(s, x_{n_j}(s))\,ds \right|$$

$$+ \left| \int_{t_0}^{t} f(s, x_{n_j}(s))\,ds - \int_{t_0}^{t} f(s, x(s))\,ds \right|$$

$$\leq \int_{t_0}^{t} |p_{n_j}(s, x_{n_j}(s)) - f(s, x_{n_j}(s))|\,ds$$

$$+ \int_{t_0}^{t} |f(s, x_{n_j}(s)) - f(s, x(s))|\,ds$$

$$\leq \varepsilon \cdot a' + \varepsilon \cdot a' = 2\varepsilon a',$$

This shows that

$$\int_{t_0}^{t} p_{n_j}(s, x_{n_j}(s))\,ds \to \int_{t_0}^{t} f(s, x(s))\,ds \quad \text{as } j \to +\infty.$$

This completes the proof of Claim 3 and hence the theorem. #

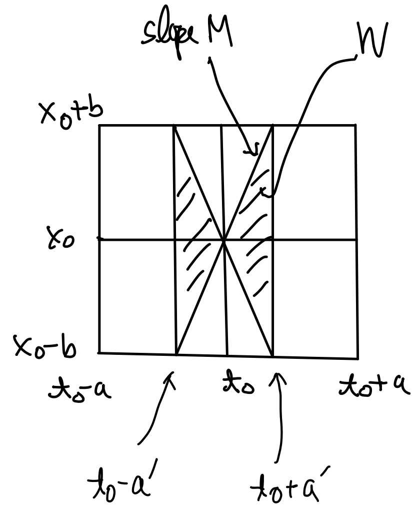
Another approach to Cauchy-Peano Theorem using Ascoli's Theorem

(Piecewise Linear Approximation)

Let $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$

$M = \sup_R |f(t,x)|$ as before.

(May assume $M \geq 1$ as we only need an upper bd)



Define

$$W = \{(t,x) \in R : |x - x_0| \leq M|t - t_0|\}$$

By symmetry,

$\mathrm{proj}(W)$ onto t-axis is $[t_0 - a', t_0 + a']$ for some $a' \in (0, a]$.

Note that $f \in C(R) \Rightarrow f \in C(W)$

$\Rightarrow$ f is uniformly continuous on W (since W is closed & bounded)

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall (t_1, x_1), (t_2, x_2) \in W$

with $|t_1 - t_2| < \delta$ and $|x_1 - x_2| < \delta$,

we have $|f(t_2, x_2) - f(t_1, x_1)| < \varepsilon$.

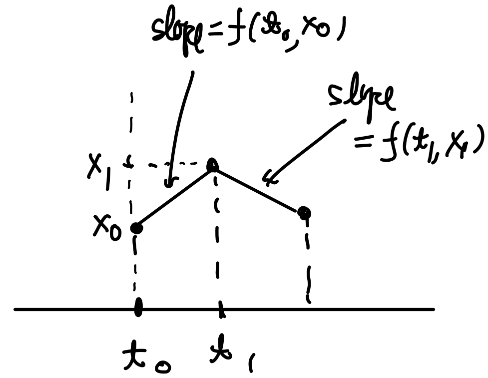
On the (half) interval $[t_0, t_0+a']$, choose

$$t_0 < t_1 < t_2 < \cdots < t_k = t_0 + a'$$

with $\quad |t_i - t_{i-1}| < \frac{\delta}{M} \quad$ for $i = 1, \cdots k$

Define a function $k_\varepsilon(t)$ on $[t_0, t_0+a']$

(1) $k_\varepsilon(t_0) = x_0$,



(2) $k_\varepsilon \big|_{[t_{i-1}, t_i]}$ is linear

with slope $f(t_{i-1}, x_{i-1})$

where x_i can be determined succesively by :

(i) x_1 determined by $k_\varepsilon \big|_{[t_0, t_1]}$ is linear, its graph passing

through (t_0, x_0) and with slope $f(t_0, x_0)$.

(ii) Note that $|f(t_0, x_0)| \le M$, $|x_1 - x_0| \le M|t_1 - t_0|$.

$\therefore$ $(t_1, x_1) \in W \subset R$ and hence $f(t_1, x_1)$ well-defined.

(iii) then x_2 determined by $k_\varepsilon \big|_{[t_1, t_2]}$ is linear, its graph

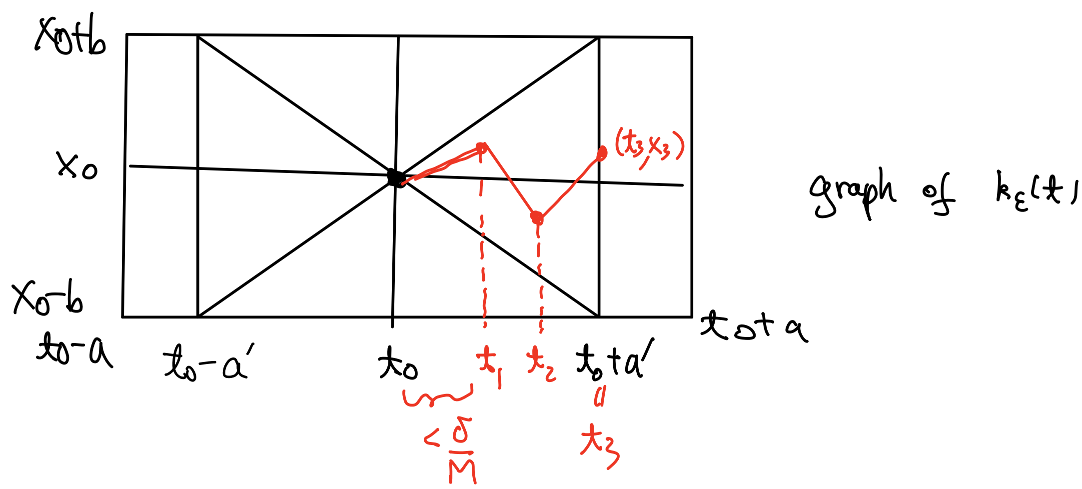
passing through (t_1, x_1) and with slope $f(t_1, x_1)$.

(ii) Similarly, $|f(t_0,x_0)|$ & $|f(t_1,x_1)| \leq M$, we have

$$|x_2 - x_0| \leq M\,|t_2 - t_0|$$

$\therefore$ $(t_2, x_2) \in W \subset R$ and $f(t_2, x_2)$ well-defined.

And so on, the function $k_\varepsilon(t)$ is defined on $[t_0, t_0+a']$



Note that

(1) k_ε is piecewise linear,

(2) $|k_\varepsilon(t) - k_\varepsilon(s)| \leq M\,|t-s|$, $\forall\, t,s \in [t_0, t_0+a']$

(By slopes $|f(t_i,x_i)| \leq M$ on each subinterval.)

$\therefore \{k_\varepsilon\}$ is equicontinuous (as subset of $C[t_0, t_0+a']$)

(3) $\{k_\varepsilon\}$ is also uniformly bounded $[t_0, t_0+a']$.

In fact, W is convex and the

ends points (t_i, x_i) with $x_i = k_\varepsilon(t_i)$ belongs to W,

we have $(t, k_\varepsilon(t)) \in W$ by piecewise linearity.

As $W \subset R$, $|k_\varepsilon(t) - x_0| \le b$ and hence

$$|k_\varepsilon(t)| \le |x_0| + b, \quad \forall t \in [t_0, t_0 + a'] \text{ and } \forall \varepsilon > 0$$

Hence Ascoli's Theorem implies that $\{k_\varepsilon\}$ is precompact.

In particular, the sequence $\{k_{\frac{1}{n}}\}_{n=1}^{\infty}$ has a convergent subsequence $\{k_{\frac{1}{n_\ell}}\}$ in $C[t_0, t_0 + a']$ with

$$k_{\frac{1}{n_\ell}}(t) \to k(t) \in C[t_0, t_0 + a'], \text{ as } \ell \to +\infty.$$

To show $k(t)$ satisfies the differential equation, we first show that k_ε is an approximated solution (including $\varepsilon = \frac{1}{n_\ell} > 0$)

For this $\varepsilon > 0$, let $\delta > 0$ be the corresponding quantity for uniform continuity of f, and t_i as in the construction of $k_\varepsilon(x)$.

Consider $t \in [t_0, t_0+a']$ and $t \neq t_i$, $i=0,1,\cdots,k-1$

Then $\exists j=1,2,\cdots k$ such that $t_{j-1} < t < t_j$.

Using $|t-t_{j-1}| < |t_j - t_{j-1}| < \frac{\delta}{M}$, we have

$$|k_\varepsilon(t) - k_\varepsilon(t_{j-1})| \le M|t-t_{j-1}| < \delta,$$

Hence
$$|f(t_{j-1}, k_\varepsilon(t_{j-1})) - f(t, k_\varepsilon(t))| < \varepsilon$$

Since k_ε is piecewise linear,

$$k_\varepsilon'(t) = f(t_{j-1}, k_\varepsilon(t_{j-1})) \qquad \text{(by our construction)}$$

Hence
$$|k_\varepsilon'(t) - f(t, k_\varepsilon(t))| < \varepsilon, \quad \forall t \in [t_0, t_0+a'] \setminus \{t_0, t_1, \cdots, t_k\}.$$

As $k_\varepsilon(t_0) = x_0$, $k_\varepsilon(t)$ is an approximated solution to

$$(\text{IVP}) \begin{cases} \dfrac{dx}{dt} = f(t,x) & \text{on } [t_0, t_0+a'] \\ x(t_0) = x_0 \end{cases}$$

in the sense that
$$\begin{cases} \dfrac{dk_\varepsilon}{dt} = f(t, k_\varepsilon) + \text{remainder} & \text{(except finitely many points)} \\ x(t_0) = x_0 \end{cases}$$

with $\|\text{remainder}\|_\infty < \varepsilon$.

Integrating the ODE, we have

$$\Rightarrow \quad k_\varepsilon(t) = k_\varepsilon(t_0) + \sum_{i=1}^{j-1} \int_{t_{i-1}}^{t_i} k_\varepsilon'(s)\,ds + \int_{t_{j-1}}^{t} k_\varepsilon'(s)\,ds$$

$$= x_0 + \int_{t_0}^{t} k_\varepsilon'(s)\,ds$$

$$\Rightarrow \left| k_\varepsilon(t) - x_0 - \int_{t_0}^{t} f(s, k_\varepsilon(s))\,ds \right| \le \int_{t_0}^{t} \left| k_\varepsilon'(s) - f(s, k_\varepsilon(s)) \right| ds < \varepsilon a'.$$

In particular, if we denote $g_\ell = k_{\frac{1}{n_\ell}}$, (ie $\varepsilon = \frac{1}{n_\ell} \to 0$),

then
$$\left| g_\ell(t) - x_0 - \int_{t_0}^{t} f(s, g_\ell(s))\,ds \right| \le \frac{a'}{n_\ell}, \quad \forall \ell = 1, 2, 3, \cdots$$

Hence

$$\left| k(t) - x_0 - \int_{t_0}^{t} f(s, k(s))\,ds \right|$$

$$\le \left| k(t) - x_0 - \int_{t_0}^{t} f(s, k(s))\,ds - g_\ell(t) + x_0 + \int_{t_0}^{t} f(s, g_\ell(s))\,ds \right|$$

$$+ \left| g_\ell(t) - x_0 - \int_{t_0}^{t} f(s, g_\ell(s))\,ds \right|$$

$$\le \| k - g_\ell \|_\infty + \int_{t_0}^{t} \left| f(s, g_\ell(s)) - f(s, k(s)) \right| ds + \frac{a'}{n_\ell}.$$

Since $\|g_\ell - k\|_\infty \to 0$ and f is uniform continuity,

$$\int_{t_0}^{t} |f(s, g_\ell(s)) - f(s, k(s))|\, ds \to 0 \quad \text{as } j \to +\infty.$$

Therefore by letting $\ell \to +\infty$, we have

$$k(t) = x_0 + \int_{t_0}^{t} f(s, k(s))\, ds, \quad \forall\, t \in [t_0, t_0 + a'].$$

$$\Rightarrow \quad \begin{cases} \dfrac{dk}{dt} = f(t, k(t)) & \forall\, t \in [t_0, t_0 + a'] \\ k(t_0) = x_0. \end{cases}$$

Similarly argument $\Rightarrow$ $\exists\, \tilde{k}$ on $t \in [t_0 - a', t_0]$

satisfying $\begin{cases} \dfrac{d\tilde{k}}{dt} = f(t, \tilde{k}(t)) & \forall\, t \in [t_0 - a', t_0] \\ \tilde{k}(t_0) = x_0. \end{cases}$

Note that by construction

$$\frac{dk}{dt}(t_0) = f(t_0, x_0) = \frac{d\tilde{k}}{dt}(t_0).$$

Hence $x(t) = \begin{cases} k(t), & t \in [t_0, t_0 + a'] \\ \tilde{k}(t), & t \in [t_0 - a', t_0] \end{cases}$

is $C^1[t_0 - a', t_0 + a']$ and solve the (IVP). #

Remarks

(i) This proof doesn't need the Picard-Lindelöf Theorem.

(ii) The spirit of this proof is more in line with solving the (IVP) numerically.

(iii) The 1st proof solve "approximated problems";
the 2nd proof solve the (original) problem "approximately".