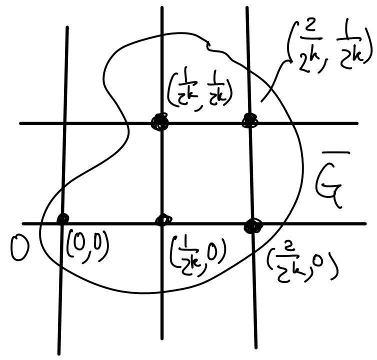
Pf: Define $E = \bigcup_{k=0}^{\infty} E_k$, where



$$E_k = \left\{ x = \frac{1}{2^k}\begin{pmatrix} l_1 \\ \vdots \\ l_m \end{pmatrix} \in \overline{G} : l_i \in \mathbb{Z},\ i = 1, \cdots, m \right\}.$$

$\overline{G}$ closed and bounded

$\Rightarrow$ E_k is finite.

Hence $E = \bigcup_k E_k$ is countable.

Let $\{f_n\}$ be a sequence in $\mathcal{E}$.

$\mathcal{E}$ bounded $\Rightarrow$

$$\exists M > 0 \text{ such that } \|f_n\|_\infty \leq M,\ \forall n$$

i.e. $|f_n(x)| \leq M$, $\forall n$ & $\forall x \in \overline{G}$

In particular, $\forall x \in E$,

$$|f_n(x)| \leq M,\ \forall n.$$

If we arrange the points of E in a sequence

$E = \{z_j\}_{j=1}^{\infty}$, then $\forall j \geq 1$,

$$\{f_n(z_j)\} \text{ is a bounded sequence.}$$

Hence one can apply Lemma 4.3 to find a subsequence

$\{g_n\}$ of $\{f_n\}$ (using the same notation "n" for the index)

such that $\forall x \in E$, $g_n(x)$ is convergent.

We claim that g_n is the required convergent subsequence

of f_n in the metric space $(C(\bar{G}), d_\infty)$.

(Note that we only have pointwise convergence for countable many points at this moment.)

Since $(C(\bar{G}), d_\infty)$ is complete, we only need to show that

$\{g_n\}$ is a Cauchy sequence in $(C(\bar{G}), d_\infty)$.

By equicontinuity of $\mathcal{E}$, ($\Rightarrow$ equicontinuity of $\{g_n\}$)

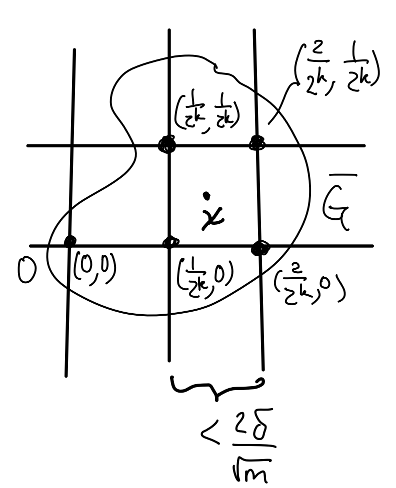
$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$|g_n(x) - g_n(y)| < \frac{\varepsilon}{3}$, $\forall n$ & $\forall x, y \in \bar{G}$ with $|x-y| < \delta$.

Note that if k satisfies $\frac{1}{2^k} < \frac{2\delta}{\sqrt{m}}$,

then $\forall x \in \overline{G}$, $\exists\ z_j \in E_k$ such that

$|x - z_j| < \delta$. (See figure)



Hence $\quad |g_n(x) - g_n(z_j)| < \frac{\varepsilon}{3}, \ \forall n$

Therefore,

$$|g_n(x) - g_m(x)| \le |g_n(x) - g_n(z_j)| + |g_n(z_j) - g_m(z_j)| + |g_m(z_j) - g_m(x)|$$

$$< \frac{2\varepsilon}{3} + |g_n(z_j) - g_m(z_j)|.$$

Since $\{g_n(z_j)\}$ is convergent, $\exists\ n_0 = n_0(z_j) \geq 0$ s.t.

$$|g_n(z_j) - g_m(z_j)| < \frac{\varepsilon}{3}, \quad \forall n, m \geq n_0(z_j).$$

$\Rightarrow \ |g_n(x) - g_m(x)| < \varepsilon, \ \forall n, m \geq n_0(z_j)$. ($z_j$ depends on x)

Now take $N_0 = \max\limits_{z_j \in E_k} n_0(z_j) \geq 0$,

E_k ← finite set

then $\forall\, x \in \overline{G}$, we have

$$|g_n(x) - g_m(x)| < \varepsilon, \quad \forall\, n, m \geq N_0.$$

i.e. $$\|g_n - g_m\|_\infty < \varepsilon, \quad \forall\, n, m \geq N_0.$$

This completes the proof of the Theorem. #