(iv) A reason for studying $C_b(X)$ instead of $C(X)$ is the fact that $C(X)$ may contain unbounded function and supnorm $\|\cdot\|_\infty$ doesn't define.

(eg: $X = \mathbb{R} = (-\infty, +\infty)$.)

However, in some cases, it is still possible to define a metrix on $C(X)$.

eg $X = \mathbb{R}^m$, $\overline{B_n(0)} = \{x \in \mathbb{R}^m : |x| \leq n\}$, $\forall n = 1, 2, 3, \cdots$

$\forall f \in C(\mathbb{R}^m)$, define

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_{\infty, \overline{B_n(0)}}}{1 + \|f-g\|_{\infty, \overline{B_n(0)}}}$$

where $\|\cdot\|_{\infty, \overline{B_n(0)}}$ is the supnorm on the closed ball $\overline{B_n(0)}$.

Then d is a complete metric on $C(\mathbb{R}^n)$. (Ex!)

(v) Recall:

> **Bolzano-Weierstrass Theorem (in $\mathbb{R}^n$):**
> Every bounded sequence (set) has (contains) a convergent subsequence (sequence).

- $C_b(X)$ may not have Bolzano-Weierstrass property.

eg. $C_b([0,1]) = C[0,1]$. Let $f_n(x) = x^n$, $x \in [0,1]$.

Then $\|f_n\|_\infty = 1$, $\forall n$.

Note that pointwise limit $f_n(x) \to \begin{cases} 1, & x=1 \\ 0, & \text{otherwise} \end{cases}$.

$\Rightarrow$ no subsequence converges in $C_b([0,1])$.

In view of note (v), we need further condition to help us to find convergence sequence in subset of $C_b(X)$.

Def: Let (X, d) be a metric space. A set $E \subset X$ is called a precompact set if every sequence in E contains a convergent subsequence.

(with limit in X, not necessary in E)

If further required that the limit belongs to E, then it is called compact.

Note: Compact set is a closed precompact set.

Pf: Let $\{x_n\} \subset E$.

$$E \text{ precompact} \Rightarrow \exists\, x_{n_j} \to z \in X$$

$$E \text{ closed} \Rightarrow z \in E$$

Hence closed precompact $\Rightarrow$ compact.

The other direction:

"compact $\Rightarrow$ closed precompact"

is trivial. #

eg: Bolzano-Weierstrass $\Rightarrow$

"$E \subset \mathbb{R}^n$ is precompact $\Leftrightarrow$ E is bounded." (Ex!)

Hence

"$E \subset \mathbb{R}^n$ is compact $\Leftrightarrow$ E is closed & bounded."

> Def: Let (X, d) be a metric space.
> A subset $\mathcal{E}$ of $C(X)$ is equicontinuous
> if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that
> $|f(x) - f(y)| < \varepsilon$, $\forall f \in \mathcal{E}$ & $d(x, y) < \delta$ $(x, y \in X)$.

Note: Clearly if $\mathcal{E}$ is equicontinuous, then

any $\mathcal{E}' \subset \mathcal{E}$ is equicontinuous.

Eg: If $X = \overline{G} \subset \mathbb{R}^n$, $G \neq \emptyset$ open & bounded.

Then $f \in C(\overline{G})$ is always uniformly continuous:

$\forall \varepsilon > 0, \ \exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon, \quad \forall \ d_{\mathbb{R}^n}(x,y) = |x-y| < \delta \qquad (x, y \in \bar{G})$$

The δ here usually depends on f.

Comparing to the definition of equicontinuity,

$\mathcal{C}$ is equicontinuous, if we can find a

<u>uniform $\delta > 0$ for all functions $f \in \mathcal{C}$</u>,

i.e. δ is <u>independent</u> of points $x, y \in \bar{G}$

and functions $f \in \mathcal{C}$.

<u>eg</u>: A function f defined on a subset $\bar{G}$ of $\mathbb{R}^n$ ($G \neq \emptyset$, open & bounded)

is called

<u>Hölder continuous</u>

if $\exists \ \alpha \in (0,1)$ such that

$$(*) \quad |f(x) - f(y)| \leq L \, |x-y|^{\alpha}, \quad \forall x, y \in \bar{G},$$

for some constant L.

- The number α is called the <u>Hölder exponent</u>.
- The function is called <u>Lipschitz</u> continuous if $(*)$ holds for $\alpha = 1$.
- For a fixed $\alpha \in (0,1]$ & $L > 0$, the family

$$\mathcal{E} = \{ f \in C(\overline{G}) : f \text{ Hölder/Lip. with exponent } \alpha \text{ and } L > 0 \}$$

is an equicontinuous family.

<u>Pf</u>: $\forall \varepsilon > 0$, let $\delta > 0$ such that $L\delta^{\alpha} < \varepsilon$.

Then $\forall f \in \mathcal{E}$, $\forall x, y \in \overline{G}$ with $|x-y| < \delta$,

$$|f(x) - f(y)| \leq L|x-y|^{\alpha} < L\delta^{\alpha} < \varepsilon. \quad \#$$

Prop 4.1 : Let $\mathcal{C}$ be a subset $C(\bar{G})$ where

$\bar{G}$ is nonempty convex in $\mathbb{R}^n$ (with G open & bounded).

Suppose that each function in $\mathcal{C}$ is differentiable and there is a uniform bound on their partial derivatives.

Then $\mathcal{C}$ is equicontinuous.

(i.e. $\mathcal{C} = \{ f \in C(\bar{G}) : f \text{ differentiable}, \left\| \frac{\partial f}{\partial x_i} \right\|_\infty \le M, \ \forall i \}$

(for same M indep. of f)

is equicontinuous provided $\bar{G}$ is convex.)

Pf: $\forall\, x, y \in \bar{G}$,

$$\bar{G} \text{ convex} \Rightarrow x + t(y-x) \in \bar{G}, \quad \forall t \in [0,1].$$

Then $f(y) - f(x) = \int_0^1 \frac{d}{dt} f(x + t(y-x))\, dt$

$$= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x + t(y-x))\,(y_i - x_i)\, dt$$

$$= \sum_{i=1}^{n} \left(\int_0^1 \frac{\partial f}{\partial x_i}(x+t(y-x))\,dt \right)(y_i - x_i)$$

$$\leq \sqrt{\sum_{i=1}^{n} \left| \int_0^1 \frac{\partial f}{\partial x_i}(x+t(y-x))\,dt \right|^2}\ |y-x|$$

$$\leq \sqrt{n}\, M\, |y-x| ,$$

where M = uniform bd. on the partial derivatives.

Interchanging x, y, we have

$$|f(y) - f(x)| \leq \sqrt{n}\, M\, |y-x| , \quad \forall\ x, y \in \overline{G}$$

Then by the above example, $\mathcal{E}$ is equicontinuous. ※

Eg 4.1 (Equicontinuous, but unbounded)

let $X = [-1, 1]$ and consider

$$\mathcal{E} = \{ x \in C[-1,1] : x'(t) = t ,\ t \in [-1,1] \} .$$

$\forall\ x \in \mathcal{E}$, $|x(t) - x(s)| \leq \|x'\|_\infty |t-s| \leq |t-s|$.

$\Rightarrow$ $\mathcal{E}$ is equicontinuous (as above).

But $\mathcal{E}$ is unbounded (in $C[-1,1]$):

$$x_n(t) = \frac{t^2}{2} + n \in \mathcal{E} \text{ has}$$

$$\|x_n\|_\infty = \frac{1}{2} + n \to +\infty \quad \text{as } n \to +\infty.$$

(Clearly, $\{x_n\}$ has no convergent subsequence.)

<u>eg 4.5</u> (Closed & Bounded, but not Equicontinuous)

Let $\mathcal{B} = \{f \in C[0,1] : |f(x)| \le 1, \forall x \in [0,1]\}$

$$\left(= \overline{B_1^\infty(0)} \right)$$

Then $\mathcal{B}$ is closed and bounded.

To show that $\mathcal{B}$ is not equicontinuous, we only need to find a subset of $\mathcal{B}$ which is not equicontinuous.

Let $\{f_n(x) = \sin nx\}_{n=1}^\infty \subset \mathcal{B}$.

<u>Claim</u>: $\{f_n(x) = \sin nx\}_{n=1}^\infty$ is not equicontinuous

Pf Suppose on the contrary that

$$\{f_n(x) = \sin nx\}_{n=1}^{\infty} \text{ is equicontinuous.}$$

Then for $\varepsilon = \frac{1}{2}$, $\exists\, \delta > 0$ such that

$\forall\, n \geq 1$, & $x, y \in [0,1]$ with $|x-y| < \delta$, we have

$$|\sin nx - \sin ny| < \frac{1}{2}.$$

However, for any $\delta > 0$, if $n > \max\{\frac{\pi}{2\delta}, \frac{\pi}{2}\}$,

we have $x = 0$ & $y = \frac{\pi}{2n} \in [0,1]$ with $|x-y| < \delta$

and $\left|\sin n \cdot 0 - \sin n \cdot \frac{\pi}{2n}\right| = |0-1| = 1 > \frac{1}{2}$

Which is a contradiction.

$\therefore$ $\{\sin nx\}_{n=1}^{\infty}$ is not equicontinuous. #

Lemma 4.3 Let $A = \{z_j\}_{j=1}^{\infty}$ be a <u>countable set</u> and

$f_n : A \to \mathbb{R}$, $n = 1, 2, \cdots$, be a sequence of functions

defined on A.

Suppose that for each $z_j \in A$,

$\{f_n(z_j)\}_{n=1}^{\infty}$ is a <u>bounded sequence</u> in $\mathbb{R}$.

Then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$

such that $\forall z_j \in A$,

$$\{f_{n_k}(z_j)\} \text{ is convergent.}$$

<u>Pf</u>: Since $f_n(z_1)$ is a bounded sequence (in $\mathbb{R}$),

$\exists$ a subsequence f_n^1 such that

$f_n^1(z_1)$ is convergent.

(Note that we have used the same index n to denote the subsequence f_{n_k}. The superscript 1 is to denote that it is convergent when evaluated at z_1.)

For this subsequence f_n^1 (of original f_n),

$f_n^1(z_2)$ is bounded (since $\{f_n^1(z_2)\} \subset \{f_n(z_2)\}$).

Hence $\exists$ a subsequence $\{f_n^2\}$ of $\{f_n^1\}$ such that

$\{f_n^2(z_2)\}$ is convergent.

Note that since $\{f_n^1\}$ is a subseq. of $\{f_n\}$,

$\{f_n^2\}$ is also a subsequence of $\{f_n\}$.

Also, $\{f_n^2(z_1)\}$ is a subseq. of the convergent subsequence $\{f_n^1(z_1)\}$,

$\therefore$ $\{f_n^2(z_1)\}$ is also convergent.

Therefore, we've found a subseq. $\{f_n^2\}$ of $\{f_n\}$ such that

- $\{f_n^2\}$ is a subseq. of $\{f_n^1\}$, and
- $\{f_n^2(z_1)\}$ and $\{f_n^2(z_2)\}$ are convergent,

Repeating the process, one can obtain sequences

$$\{f_n^j\} \quad (j=0,1,2,\cdots \text{ with } f_n^0 = f_n)$$

such that

(i) $\{f_n^{j+1}\}$ is a subsequence of $\{f_n^j\}$, $\forall j=0,1,2\cdots$

(ii) $\{f_n^j(z_1)\}, \{f_n^j(z_2)\}, \cdots, \{f_n^j(z_j)\}$ are convergent $(j\geq 1)$

$$\begin{array}{cccccc|c}
 & & & & & & \text{convergent at} \\
f_1^1 & f_2^1 & f_3^1 & \cdots & f_n^1 & \cdots & z_1 \\
f_1^2 & f_2^2 & f_3^2 & \cdots & f_n^2 & \cdots & z_1, z_2 \\
f_1^3 & f_2^3 & f_3^3 & \cdots & f_n^3 & \cdots & z_1, z_2, z_3 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
f_1^n & f_2^n & f_3^n & \cdots & f_n^n & \cdots & z_1, z_2, \cdots, z_n \\
\vdots & \vdots & \vdots & & \vdots & & \vdots
\end{array}$$

Define $g_n = f_n^n$, $\forall n\geq 1$. (the diagonal sequence),

then $\{g_n\}$ is a subsequence of $\{f_n\}$ and

for any fixed $j=1,2,\cdots$, $\quad g_n(z_j) = f_n^n(z_j)$

As $n \to \infty$, $n \geq j$ for sufficiently large n.

Hence $\{f_n^n(z_j)\}$ is a subsequence of the convergent sequence

$\{f_n^j(z_j)\}$ for all sufficiently large n.

Therefore $\{g_n(z_j)\}$ is convergent.

This completes the proof of the Lemma. #

(This method of finding g_n is called the Cantor's diagonal trick.)

Thm 4.2 (Ascoli's Theorem)

Suppose that G is a bounded nonempty open set in $\mathbb{R}^m$.

Then a set $\mathcal{E} \subset C(\bar{G})\ (= C_b(\bar{G}))$ is precompact

iff $\mathcal{E}$ is bounded (in supnorm) and equicontinuous.