Ch4 Space of Continuous Functions

§4.1 Ascoli's Theorem

<u>Notation</u>: If (X, d) = metric space, we denote

$$C_b(X) = \{ f \in C(X) : |f(x)| \leq M, \ \forall x \in X, \text{ for some } M \}$$

the vector space of all <u>bounded continuous functions</u> on X.

Clearly,

$$C_b(X) \subset C(X).$$

$$(C(X) = \text{set of continuous functions on } X.)$$

<u>eg</u>: If G = (nonempty) bounded open set in $\mathbb{R}^n$, then

$$C_b(\overline{G}) = C(\overline{G})$$

as $\overline{G}$ is closed and bounded, $f \in C(\overline{G})$ has to be bounded.

<u>Recall</u>: A norm $\|\cdot\|$ on a real vector space X is defined by the following properties:

(N1) $\|x\| \geq 0$ & "$\|x\| = 0 \Leftrightarrow x = 0$"

(N2) $\|\alpha x\| = |\alpha| \, \|x\|$ $\quad (\alpha \in \mathbb{R})$

(N3) $\|x+y\| \leq \|x\| + \|y\|$.

And a vector space with norm $(X, \|\cdot\|)$ is called a norm space. A norm space has a natural metric

$$d(x,y) = \|x-y\|.$$

<u>Fact</u>: The <u>supnorm</u> $$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

is a norm on $C_b(X)$.

And we always assume $C_b(X)$ with metric

$$d_\infty(f,g) = \|f-g\|_\infty.$$

given by the <u>supnorm</u>.

Similar to $(C[a,b], d_\infty)$, we have

$\boxed{\text{Prop: } (C_b(X), d_\infty) \text{ is complete (for any metric space } (X, d))}$

Pf: Let $\{f_n\}$ be a Cauchy seq. in $(C_b(X), d_\infty)$

Then $\forall \varepsilon > 0$, $\exists n_0 \geq 0$ s.t.

$$\| f_m - f_n \|_\infty < \frac{\varepsilon}{4}, \quad \forall m, n \geq n_0.$$

In particular, $\forall x \in X$,

$$(*)_1 \qquad |f_m(x) - f_n(x)| \leq \| f_m - f_n \|_\infty < \frac{\varepsilon}{4}, \forall m, n \geq n_0$$

$\Rightarrow$ $\{f_n(x)\}$ is a Cauchy seq. in $\mathbb{R}$.

By completeness of $\mathbb{R}$ (not X),

$$\lim_{n\to\infty} f_n(x) \quad \text{exists.}$$

In general, it depends on x. Let denote it by

$$f(x) = \lim_{n\to\infty} f_n(x), \quad \forall x \in X.$$

This gives a function f on X.

Claim 1 f is bounded.

Pf: Letting $m \to \infty$ in $(*)_1$, we have

$\forall \varepsilon > 0$, and $\forall x \in X$,

$$(*)_2 \quad |f(x) - f_n(x)| \leq \frac{\varepsilon}{4}, \quad \forall n \geq n_0$$

In particular, $|f(x) - f_{n_0}(x)| \leq \frac{\varepsilon}{4}$, $\forall \varepsilon > 0$, $\forall x \in X$.

$\Rightarrow \forall x \in X$, $|f(x)| \leq \frac{\varepsilon}{4} + |f_{n_0}(x)| \leq \frac{\varepsilon}{4} + M_0$,

where M_0 is a bound for f_{n_0}.

$\therefore$ f is bounded.

Claim 2 : f is continuous.

Pf: f_{n_0} cts $\Rightarrow \forall x_0 \in X$ & $\forall \varepsilon > 0$, $\exists \delta > 0$

s.t. $|f_{n_0}(x) - f_{n_0}(x_0)| < \frac{\varepsilon}{4}$, $\forall d(x, x_0) < \delta$.

Then together with $(*)_2$,

$$|f(x) - f(x_0)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)|$$

$$\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon, \qquad \forall d(x, x_0) < \delta.$$

$\therefore f$ is cts at x_0.

Since $x_0 \in X$ is arbitrary, f is cts on X.

<u>Claims 1 & 2</u> $\Rightarrow f \in C_b(X)$.

Finally, by $(*)_2$, $\sup_{x \in X} |f(x) - f_n(x)| \leq \frac{\varepsilon}{4}$, $\forall n \geq n_0$

i.e. $\quad d_\infty(f_n, f) \leq \frac{\varepsilon}{4}, \quad \forall n \geq n_0$

So $\quad d_\infty(f_n, f) \to 0 \quad$ as $n \to \infty$

That is $\quad f_n \to f$ in $(C_b(X), d_\infty)$. #

<u>Notes</u>:

(i) We've just proved that $(C_b(X), d_\infty)$ is a <u>Banach space</u>, i.e. a <u>complete</u> normed vector space.

(ii) $C_b(X)$ is <u>usually</u> of <u>infinite dimensional</u>:

<u>egs</u>: When $X = \mathbb{R}^n$ or subset with non-empty interior in $\mathbb{R}^n$

Explicit eg: $\mathbb{X} = [0,1] \subset \mathbb{R}$, then

$$\{f_n(x) = x^n\}_{n=0}^{\infty} \subset C_b(\mathbb{X}) .$$

Clearly, $\{x^n\}_{n=0}^{\infty}$ is a linearly indep. subset.

$\Rightarrow$ $C_b(\mathbb{X}) = C[0,1]$ is of infinite dimensional.

(iii) $C_b(\mathbb{X})$ could be of finite dimension:

eg: $\mathbb{X} = \{p_1, \cdots, p_n\}$ finite set with discrete metric

Then
$$\begin{array}{ccc} \mathbb{X} & \to & \mathbb{R}^n \\ \psi & & \psi \\ f & \mapsto & (f(p_1), \cdots, f(p_n)) \end{array}$$
is a linear bijection.