<u>General Case</u>:

Consider $\quad \tilde{F}(x) = (DF)^{-1}(x_0)\left[F(x+x_0) - y_0\right]$

Then $\quad \tilde{F}(0) = 0$

$\tilde{F}$ defined on an open set

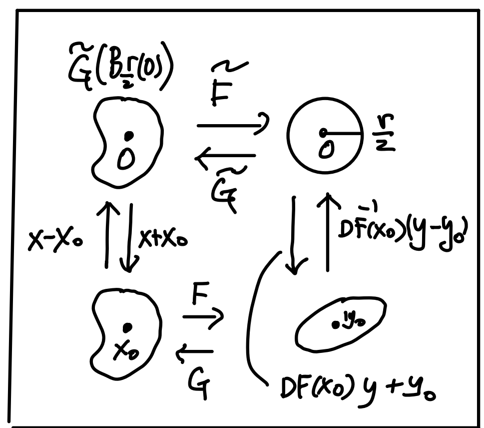
$\tilde{U} = U - x_0 = \{x : x + x_0 \in U\}$ with $0 \in \tilde{U}$.

and $\quad D\tilde{F}(0) = (DF)^{-1}(x_0)\, DF(x_0) = I$.

By the special case, $\exists\, r > 0$ such that

$$\exists\ \tilde{G} : B_{\frac{r}{2}}(0) \longrightarrow \tilde{G}\left(B_{\frac{r}{2}}(0)\right) \subset B_r(0) \subset \tilde{U}$$

s.t. $\tilde{G}$ is the local inverse of $\tilde{F}$.



Let $\quad W = DF(x_0)\left(B_{\frac{r}{2}}(0)\right) + y_0$

$V = \tilde{G}\left(B_{\frac{r}{2}}(0)\right) + x_0$, (then $V \subset U$ & $x_0 \in V$)

and $\quad G : W \to V$ by

$$G(y) = \tilde{G}\left((DF)^{-1}(x_0)\,(y - y_0)\right) + x_0, \quad \forall y \in W.$$

Clearly G maps W bijective onto V.

Since $\tilde{F}(x) = (DF)^{-1}(x_0)\left[F(x+x_0) - y_0\right]$,

we have $\quad F(x+x_0) = (DF)(x_0)\tilde{F}(x) + y_0, \quad \forall x \in B_r(0)$

$$\Rightarrow \quad F(x) = (DF)(x_0)\tilde{F}(x-x_0) + y_0, \quad x \in V$$

Hence $\forall y \in W$

$$F(G(y)) = (DF)(x_0)\tilde{F}(G(y) - x_0) + y_0$$

$$= y_0 + (DF)(x_0)\tilde{F}\left[\tilde{G}\left((DF)^{-1}(x_0)(y-y_0)\right)\right] \quad \left(\text{by definition of } G\right)$$

$$= y_0 + (DF)(x_0)\left((DF)^{-1}(x_0)(y-y_0)\right) \quad \left(\text{since } \tilde{F}\circ\tilde{G} = I\right)$$

$$= y_0 + y - y_0 = y$$

$\therefore$ G is the local inverse of F

The remaining facts that $F \in C^k$ $(k \geq 1) \Rightarrow G \in C^k$ is clear from the definition of G, and the results on $\tilde{G}$ ($\&$ $\tilde{F}$) in the special case. #

Def: A C^k-map $F: V \to W$ (V, W open in $\mathbb{R}^n$) is a <u>C^k-diffeomorphism</u> if F^{-1} exists and is also C^k.

<u>Note</u>: (i) The IFT can be rephased as:

If $F: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n \in \underset{\smile}{C^k}$ and

DF is <u>nonsingular</u> at a point $x_0 \in U$,

then F is a C^k-diffeomorphism between some nbds V and W of x_0 & $F(x_0)$ respectively.

(ii) If $F: V \to W$ is a C^k-diffeomorphism, then $\forall$ function $\varphi: W \to \mathbb{R}$, there corresponds a function

$$\psi = \varphi \circ F : V \to \mathbb{R}.$$

Conversely, $\forall$ function $\psi: V \to \mathbb{R}$, there corresponds a function

$$\varphi = \psi \circ F^{-1} : W \to \mathbb{R}.$$

Moreover, <u>φ is $C^k \iff \psi$ is C^k</u>.

Thus every C^k-diffeomaphism gives rise to a <u>"local C^k-change of coordinates"</u>.

Thm 3.5 (Implicit Function Theorem)

Let U be an open set in $\mathbb{R}^n \times \mathbb{R}^m$

$F : U \to \mathbb{R}^m$ is a C^1-map.

Suppose that $(x_0, y_0) \in U$ satisfies

$F(x_0, y_0) = 0$, and $D_y F(x_0, y_0)$ is invertible in $\mathbb{R}^m$.

Then

(1) $\exists$ an open set of the form $V_1 \times V_2 \subset U$ containing (x_0, y_0) and a C^1-map

$$\varphi : \overset{\subset \mathbb{R}^n}{V_1} \longrightarrow \overset{\subset \mathbb{R}^m}{V_2} \quad \text{with} \quad \varphi(x_0) = y_0$$

such that $\boxed{F(x, \varphi(x)) = 0, \ \forall x \in V_1.}$

(2) $\varphi : V_1 \to V_2$ is C^k when F is C^k, $1 \le k \le \infty$.

(3) Moreover, assume further that $D_y F$ is invertible in $V_1 \times V_2$.

Then, if $\psi : V_1 \to V_2$ is another C^1-map satisfying

$F(x, \psi(x)) = 0$, we have $\psi \equiv \varphi$.

<u>Note</u>: If $F = \begin{pmatrix} F_1(x_1, \dots, x_n, y_1, \dots, y_m) \\ \vdots \\ F_m(x_1, \dots, x_n, y_1, \dots, y_m) \end{pmatrix}$,

then

$$D_y F = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}$$

is $m \times m$ & can be regarded as a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^m$

In general, for a map F such that

<u>$DF(x_0, y_0)$ has rank m</u>,

then one can rearrange the independent variables to make the $m \times m$ submatrix corresponding to the <u>last m columns</u> of the Jocabian matrix <u>invertible</u>, i.e. in the situation of the theorem.

Hence the condition that <u>$D_y F(x_0, y_0)$ is invertible</u> in the Implicit Function Theorem can be generalized to

$$\boxed{\text{rank } DF(x_0, y_0) = m}.$$

Pf of Implicit Function Theorem : (Using Inverse Function Theorem)

Define
$$\begin{array}{ccc} \Phi : U \subset \mathbb{R}^n \times \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \times \mathbb{R}^m \\ \Psi & & \Psi \\ (x,y) & \longmapsto & (x, F(x,y)) \end{array}$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \in \mathbb{R}^m$.

Then $\Phi(x_0, y_0) = (x_0, 0)$.

Clearly Φ is C^k if F is C^k.

And
$$\Phi = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ F_1(x_1, \cdots, x_n, y_1, \cdots, y_m) \\ \vdots \\ F_m(x_1, \cdots, x_n, y_1, \cdots, y_m) \end{pmatrix}$$

$\Rightarrow$
$$D\Phi = \left(\begin{array}{ccc|ccc} 1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & 1 & & & \\ \hline \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} & \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} & \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{array}\right)$$

Since $D_y F \Big|_{(x_0, y_0)} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}$ is invertible in $\mathbb{R}^m$,

$D\Phi\big|_{(x_0,y_0)}$ is invertible in $\mathbb{R}^n \times \mathbb{R}^m$.

Applying Inverse Function Theorem, $\exists$ local C^1-inverse

$$\Psi = (\Psi_1, \Psi_2) : W^{\subset \mathbb{R}^n \times \mathbb{R}^m} \longrightarrow V^{\subset U}, \quad \text{with}$$

$$(x_0, y_0) = \Psi(x_0, 0) = (\Psi_1(x_0, 0), \Psi_2(x_0, 0)),$$

where W and V are open nbds. of

$\Phi(x_0, y_0) = (x_0, 0)$ and (x_0, y_0) respectively,

and is C^k when F is C^k.

By shrinking the nbds, we may assume

V is of the form $V_1 \times V_2$,

where V_1 open in $\mathbb{R}^n$ containing x_0;

V_2 open in $\mathbb{R}^m$ containing y_0.

Now $\forall\, (x, z) \in W$,

$$(x, z) = \Phi\,(\Psi_1(x, z), \Psi_2(x, z))$$

$$= (\Psi_1(x,z),\ F(\Psi_1(x,z),\ \Psi_2(x,z)))$$

$$\therefore \begin{cases} x = \Psi_1(x,z) \\ z = F(\Psi_1(x,z), \Psi_2(x,z)) \end{cases}$$

$$\Rightarrow \qquad z = F(x, \Psi_2(x,z))$$

In particular, we can take $z=0$ & hence

$$F(x, \Psi_2(x,0)) = 0, \qquad \forall x = \Psi_1(x,0) \in V_1.$$

$\therefore$ $\varphi : V_1 \to V_2 : x \mapsto \Psi_2(x,0)$ is the required map

s.t. $\begin{cases} \varphi(x_0) = \Psi_2(x_0,0) = y_0, \\ F(x,\varphi(x)) = 0 \end{cases}$

and is C^k when F is C^k. We've proved (1) & (2).

For (3), $D_y F$ is invertible in $V_1 \times V_2$

$$\Rightarrow \int_0^1 D_y F(x, y_1 + t(y_2 - y_1))\, dt \text{ is nonsingular}$$

for (x,y_1) & $(x,y_2) \in V_1 \times V_2$. (as one may assume V_2 is a ball)

Now if $\psi : V_1 \to V_2$ is another C^1-map s.t.

$$F(x, \psi(x)) = 0,$$

then
$$0 = F(x, \psi(x)) - F(x, \varphi(x))$$

$$= \left(\int_0^1 D_y F(x, \varphi(x) + t(\psi(x) - \varphi(x))) dt \right) (\psi(x) - \varphi(x))$$

$\int_0^1 D_y F(x, \varphi(x) + t(\psi(x) - \varphi(x))) dt$ nonsingular $\Rightarrow$

$$\psi(x) \equiv \varphi(x), \quad \forall x \in V_1. \quad \#$$

<u>Remark</u>: Implicit Function Theorem and Inverse Function Theorem are in fact equivalent:

If $F : U \to \mathbb{R}^n$ as in assumption of the Inverse Function Theorem, then define

$$\tilde{F}(x,y) : U \times \mathbb{R}^n \to \mathbb{R}^n \qquad (n+n \text{ to } n\text{-dim})$$

$$\begin{array}{c} \psi \\ (x, y) \mapsto F(x) - y \end{array} \qquad \left(\begin{array}{c} \text{i.e. } m = n \\ \text{in the theorem} \end{array} \right)$$

which is C^1.

Note that $\tilde{F}(x_0, y_0) = F(x_0) - y_0 = 0$, and

$$D_x \tilde{F}(x_0, y_0) = DF(x_0) \text{ is invertible}$$

$\Rightarrow$ $D\tilde{F}(x_0, y_0)$ is of full rank ($\text{rank } D\tilde{F}(x_0, y_0) = n$)

By Implicit Function Theorem, $\exists$ C^1-map $\varphi(y)$ near y_0 such that

$$\varphi(y_0) = x_0 \quad \text{and} \quad \tilde{F}(\varphi(y), y) = 0.$$

(Note the different in the notations)

i.e. $\quad F(\varphi(y)) - y = 0 \quad$ near y_0

$\therefore$ $x = \varphi(y)$ is the local inverse.

[Concrete examples are omitted since it should be given in advanced calculus already. A few explicit examples are given in Prof Chou's notes.]

§3.4 Picard–Lindelöf Theorem for Differential Equations

Let f be a function defined on

$$R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$$

where $(t_0, x_0) \in \mathbb{R}^2$ and $a, b > 0$.

We consider Cauchy Problem (Initial Value Problem)

$$(\text{IVP}) \begin{cases} \dfrac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

i.e. find a function $x(t)$ defined in a perhaps smaller interval

$$x : [t_0 - a', t_0 + a'] \longrightarrow [x_0 - b, x_0 + b]$$

for some $0 < a' \leq a$, such that

$$\begin{cases} x(t) \text{ is differentiable,} \\ x(t_0) = x_0, \quad \text{and} \\ \dfrac{dx}{dt}(t) = f(t, x(t)), \quad \forall t \in [t_0 - a', t_0 + a'] \end{cases}$$

eg 3.14 Consider $\begin{cases} \frac{dx}{dt} = 1+x^2 \\ x(0) = 0 \end{cases}$

Here $f(t,x) = 1+x^2$ is smooth on

$[-a,a] \times [-b,b]$ for any $a, b > 0$.

However, the solution

$$x(t) = \tan t$$

defined only on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$\therefore$ Even for nice f, we may still have $a' < a$.

Recall :

(i) f defined in $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ satisfies the

Lipschitz condition (uniform in t)

if $\exists\, L > 0$ s.t. $\forall\, (t, x_1), (t, x_2) \in R$,

$$|f(t,x_1) - f(t,x_2)| \le L |x_1 - x_2| .$$

(ii) In particular, $f(t, \cdot)$ is Lip. cts in x, $\forall\, t \in [t_0 - a, t_0 + a]$.

(iii) L is called a Lipschitz constant.

(iv) If L is a Lip. constant for f, then any $L' > L$ is also a Lip. constant.

(v) Not all cts. functions satisfy the Lip. condition.

e.g. $f(t,x) = t x^{1/2}$ is cts, but not Lip. near 0.

(vi) If $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ and

$$f(t,x) : R \to \mathbb{R} \text{ is } C^1,$$

then $f(t,x)$ satisfies the Lip. condition:

in fact, for some $y \in [x_0 - b, x_0 + b]$,

$$|f(t,x_2) - f(t,x_1)| = \left|\frac{\partial f}{\partial x}(t,y)(x_2 - x_1)\right|$$

Hence $|f(t,x_2) - f(t,x_1)| \leq L|x_2 - x_1|$,

for $L = \max\left\{\left|\frac{\partial f}{\partial x}(t,x)\right| : (t,x) \in R\right\}$.

Thm 3.10 (Picard–Lindelöf Theorem)

(Fundamental Theorem of Existence and Uniqueness of Differential Equations)

Let f be continuous function on

$$R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b], \quad ((t_0, x_0) \in \mathbb{R}^2, \; a, b > 0)$$

satisfies the Lipschitz condition on R (uniform in t).

Then $\exists \; a' \in (0, a]$ and

$x \in C^1[t_0 - a', t_0 + a']$

such that

- $x_0 - b \le x(t) \le x_0 + b, \quad \forall \, t \in [t_0 - a', t_0 + a']$ and
- solving the Cauchy Problem (IVP)

Furthermore, x is the unique solution in $[t_0 - a', t_0 + a']$.

Note: a' can be taken to be any number satisfying

$$0 < a' < \min\left\{a, \frac{b}{M}, \frac{1}{L}\right\}$$

where $M = \sup\{|f(t, x)| : (t, x) \in R\}$ &

L = Lip. const. for f.

$\boxed{\begin{array}{l}\text{Prop 3.11 : Setting as in Thm 3.10, every solution } x \text{ of (IVP)} \\ \text{from } [t_0-a', t_0+a'] \text{ to } [x_0-b, x_0+b] \text{ satisfies} \\ \text{the equation } \boxed{x(t) = x_0 + \int_{t_0}^{t} f(t, x(t))\,dt} \quad (3.7) \\ \text{Conversely, every } x(t) \in C[t_0-a', t_0+a'] \text{ satisfying (3.7)} \\ \text{is } C^1 \text{ and solves (IVP).}\end{array}}$

Pf: Obvious, by Fundamental Theorem of Calculus.

Proof of Picard-Lindelöf Theorem:

For $a'>0$ to be chosen later, we let

$x_0-b \le \varphi(t) \le x_0+b$

$$\mathbb{X} = \{\varphi \in C[t_0-a', t_0+a'] : \varphi(t_0) = x_0,\ \varphi(t) \in [x_0-b, x_0+b]\}$$

with (uniform) metric d_∞ on $\mathbb{X}$.

First note that $\mathbb{X}$ is a closed subset in the complete metric space $(C[t_0-a', t_0+a'], d_\infty)$.

Hence $(\mathbb{X}, d_\infty)$ is complete.

Define T on $\mathbb{X}$ by

$$(T\varphi)(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds$$

(This is well-defined as $\varphi(s) \in [x_0 - b, x_0 + b]$.)

Clearly

$$\begin{cases} T\varphi \in C[t_0 - a', t + a'] \ \& \\ (T\varphi)(t_0) = x_0. \end{cases}$$

To show $T\varphi \in \mathbb{X}$, we still need

$$(T\varphi)(t) \in [x_0 - b, x_0 + b].$$

Let $M = \sup_{(t,x) \in R} |f(t,x)|$.

Then $\forall\, t \in [t_0 - a', t_0 + a']$,

$$|(T\varphi)(t) - x_0| = \left| \int_{t_0}^{t} f(s, \varphi(s))\, ds \right| \le M |t - t_0|$$

$$\le M a'$$

If we choose $0 < a' \le \frac{b}{M}$, then

$$|(T\varphi)(t) - x_0| \le b \Rightarrow T\varphi \in \mathbb{X}.$$

This is, for $0 < a' \leq \frac{b}{M}$,

$$\begin{cases} T : X \to X \text{ is a self-map from a} \\ \text{complete metric space } (X, d_\infty) \text{ to itself.} \end{cases}$$

To see whether T is a contraction, we check

$$\left|(T\varphi_2 - T\varphi_1)(t)\right| = \left|\left(x_0 + \int_{t_0}^{t} f(s, \varphi_2(s))ds\right) - \left(x_0 + \int_{t_0}^{t} f(s, \varphi_1(s))ds\right)\right|$$

$$\leq \int_{t_0}^{t} \left|f(s, \varphi_2(s)) - f(s, \varphi_1(s))\right| ds$$

$$\leq L \int_{t_0}^{t} |\varphi_2(s) - \varphi_1(s)| ds \qquad \text{(by Lip. condition)}$$

$$\leq L|t - t_0| \sup_{[t_0 - a', t_0 + a']} |\varphi_2(s) - \varphi_1(s)|$$

$$\leq L a' d_\infty(\varphi_2, \varphi_1)$$

Therefore, if we further require $La' = \gamma < 1$,

then T is a contraction:

$$d_\infty(T\varphi_2, T\varphi_1) \leq \gamma d_\infty(\varphi_2, \varphi_1) \quad \text{with } \gamma = La' < 1.$$

In conclusion, if

$$0 < a' < \min\{a, \frac{b}{M}, \frac{1}{L}\},$$

then $T: \mathbb{X} \to \mathbb{X}$ is a contraction on a complete metric space $(\mathbb{X}, d_\infty)$

Therefore, by Contraction Mapping Principle,

T admits a unique fixed point $x(t) \in \mathbb{X}$.

By Prop 3.11, we've proved Thm 3.10. ※

Notes:

(1) Existence part of Picard-Lindelöf Thm still holds with $f(t,x)$ cts only (without Lip. condition)

However, the solution may not be unique:

eg: Consider $f(t,x) = |x|^{1/2}$ on $\mathbb{R}\times\mathbb{R}$. f is cts, but not Lip. cts.

(Cauchy Problem) $\begin{cases} \frac{dx}{dt} = |x|^{1/2} & \text{on } \mathbb{R} \\ x(0) = 0 \end{cases}$

has solutions • $x_1(t) \equiv 0$ and

• $x_2(t) = \begin{cases} \frac{1}{4}t^2, & t \geq 0 \\ -\frac{1}{4}t^2, & t < 0 \end{cases}$

(Check: x_2 is differentiable with $\frac{dx_2}{dt} = \frac{1}{2}|t| = |x_2|^{1/2}$, $\forall t \in \mathbb{R}$
(& $x_2(0) = 0$))

(2) Uniqueness holds regardless of the size of the interval of existence.

(Proof omitted as it is more in the curriculum of ODE.

see Prof Chou's notes for a proof.)

(3) The proof works for system of ODEs,

just the x and f become vector-valued:

Thm 3.13 (Picard-Lindelöf Theorem for Systems)

Consider (IVP) $\begin{cases} \dfrac{dx}{dt} = f(t,x) \\ x(t_0) = x_0, \end{cases}$ with $x_0 = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

where

- $x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \in [x_1-b, x_1+b] \times \cdots \times [x_n-b, x_n+b]$ and

- $f(t,x) = \begin{pmatrix} f_1(t,x) \\ \vdots \\ f_n(t,x) \end{pmatrix} \in C(R)$, with

- $R = [t_0-a, t_0+a] \times [x_1-b, x_1+b] \times \cdots \times [x_n-b, x_n+b]$,

satisfying (Lipschitz condition (uniform in t))

$$|f(t,x) - f(t,y)| \leq L|x-y|, \quad \forall (t,x), (t,y) \in R,$$

for some constant $L > 0$.

There exists a unique solution $x \in C^1[t_0-a', t_0+a']$ with

$$x(t) \in [x_1-b, x_1+b] \times \cdots \times [x_n-b, x_n+b], \quad \forall t \in [t_0-a', t_0+a']$$

to (IVP), where a' satisfies

$$0 < a' < \min\left\{a, \frac{b}{M}, \frac{1}{L}\right\},$$

here $M = \max\limits_{j=1,\cdots,n} \sup\limits_{R} |f_j(t,x)|$.

(4) The Picard-Lindelöf Theorem for system can be applied to initial value problem for higher order ordinary differential equations :

$$
(IVP) \begin{cases} \dfrac{d^m x}{dt^m} = f\left(t, x, \dfrac{dx}{dt}, \cdots, \dfrac{d^{m-1}x}{dt^{m-1}}\right) \\ x(t_0) = x_0 \\ \dfrac{dx}{dt}(t_0) = x_1 \\ \quad \vdots \\ \dfrac{d^{m-1}x}{dt^{m-1}}(t_0) = x_{m-1} \end{cases}
$$

By letting $\vec{X} = \begin{pmatrix} x \\ \frac{dx}{dt} \\ \vdots \\ \frac{d^{m-1}x}{dt^{m-1}} \end{pmatrix}$, then

$$
\frac{d\vec{X}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{d^2x}{dt^2} \\ \vdots \\ \frac{d^m x}{dt^m} \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{d^2x}{dt^2} \\ \vdots \\ f\left(t, x, \frac{dx}{dt}, \cdots, \frac{d^{m-1}x}{dt^{m-1}}\right) \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix} = \vec{f}(t, \vec{X})
$$

with $\vec{X}(t_0) = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix}$.