Proof of IFT (Thm 3.7)

Special Case: $x_0 = 0$, $y_0 = F(x_0) = F(0) = 0$,

$$DF(0) = I \quad (\text{the Identity})$$

Step 1 Let $\Psi(x) = -x + F(x)$. $\quad (F(x) = x + \Psi(x))$

Then $\exists r > 0$ s.t.

$$|\Psi(x_2) - \Psi(x_1)| \leq \frac{1}{2}|x_2 - x_1| \text{ on } \overline{B_r(0)}.$$

Pf of Step 1: As $0 \in U$ and U is open, $\exists r_0 > 0$ s.t.

$$\overline{B_{r_0}(0)} \subset U.$$

Then $\forall x_1, x_2 \in \overline{B_{r_0}(0)}$

$$\Psi(x_1) - \Psi(x_2) = -x_1 + F(x_1) + x_2 - F(x_2)$$

(prop. 3.6)
$$= \left(\int_0^1 DF(x_2 + t(x_1 - x_2))\,dt\right)(x_1 - x_2) - (x_1 - x_2)$$

$$= \left[\int_0^1 DF(x_2 + t(x_1 - x_2))\,dt - I\right](x_1 - x_2).$$

$$= \int_0^1 \left[DF(x_2 + t(x_1 - x_2)) - DF(0)\right]dt\ (x_1 - x_2)$$

As F is C^1,

$\forall \varepsilon > 0$, $\exists r > 0$, $(r \leq r_0)$ such that

$$\|DF(x) - DF(0)\| < \varepsilon, \quad \forall x \in \overline{B_r(0)},$$

where $\|(b_{ij})\| = \sqrt{\sum_{i,j} b_{ij}^2}$ for any $n \times n$ matrix (b_{ij}).

Since $\overline{B_r(0)} \subset \mathbb{R}^n$ is convex,

$$x_1, x_2 \in \overline{B_r(0)} \Rightarrow x_2 + t(x_1 - x_2) \in \overline{B_r(0)}, \text{ for } t \in (0,1)$$

Hence $\forall \varepsilon > 0$, $\exists r > 0$ $(r \leq r_0)$ such that

$$\|DF(x_2 + t(x_1 - x_2)) - DF(0)\| < \varepsilon, \quad \forall x_1, x_2 \in \overline{B_r(0)} \text{ and } t \in (0,1).$$

Therefore $|\Psi(x_1) - \Psi(x_2)| \leq \varepsilon |x_1 - x_2|, \quad \forall x_1, x_2 \in \overline{B_r(0)}$

Choosing $\varepsilon = \frac{1}{2} > 0$, then $\exists r > 0$, $(r \leq r_0)$ s.t.

$$|\Psi(x_1) - \Psi(x_2)| \leq \frac{1}{2}|x_1 - x_2|, \quad \forall x_1, x_2 \in \overline{B_r(0)}.$$

This completes the proof of Step 1 of the special case. #

<u>Step 2</u> $r>0$ as in Step 1. Then

$$\forall y \in B_{\frac{r}{2}}(0), \quad \exists x \in B_r(0) \text{ such that } F(x)=y.$$

And the local inverse G of F,

$$G: B_{\frac{r}{2}}(0) \to G(B_{\frac{r}{2}}(0)) \subset B_r(0)$$

satisfies

$$|G(y_1)-G(y_2)| \leq 2|y_1-y_2|, \quad \forall y_1, y_2 \in B_{\frac{r}{2}}(0).$$

with $G(B_{\frac{r}{2}}(0))$ open in $B_r(0)$.

<u>Pf of Step 2</u>: By Step 1, one can apply Thm 3.4 (Perturbation of Identity) to show that

$$\forall y \in \overline{B_R(0)} \text{ with } R=(1-\tfrac{1}{2})\cdot r=\tfrac{r}{2}$$

$$\exists x \in \overline{B_r(0)} \quad \text{s.t. } F(x)=y.$$

Then remark (2) (after the proof of Thm 3.4)

$$\Rightarrow \forall y \in B_R(0), \ \exists x \in B_r(0) \text{ s.t. } F(x)=y.$$

and by remark (3) (after the proof of Thm 3.4)

(which is the Remark 3.1 of Prof Chou's notes).

we have

$$|G(y_1)-G(y_2)| \le \frac{1}{1-\frac{1}{2}}|y_1-y_2|$$

$$= 2|y_1-y_2| \quad \forall y_1, y_2 \in \overline{B_{\frac{r}{2}}(0)}.$$

Finally, remark (2) again

$\Rightarrow$ $G(B_{\frac{r}{2}}(0))$ is open in $B_r(0)$. #

<u>Step3</u> G is differentiable on $B_{\frac{r}{2}}(0)$ and

$$DG(y) = (DF)^{-1}(G(y)), \quad \forall y \in B_{\frac{r}{2}}(0).$$

<u>Pf of Step3</u>: As $DF(0)=I$ & $F \in C^1$, we may assume that

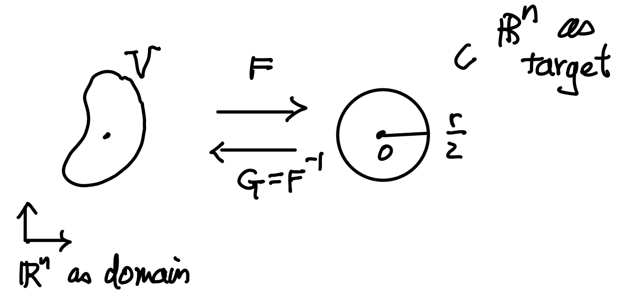
$DF(x)$ is invertible $\forall x \in B_r(0)$ for the $r>0$

given in step 1

(Since we may always choose a smaller r in the proof of Step 1.)

Let $W = B_{\frac{r}{2}}(0)$ $\Big(= \underset{\uparrow \text{ as in step 2}}{B_R(0)}\Big)$ and

$V = G(B_{\frac{r}{2}}(0)) = G(W) \ni 0$.



Then $G: W \to V$ (and $F: V \to W$)

For $y_1 \in W = B_{\frac{r}{2}}(0)$ &

$y_1 + y \in W = B_{\frac{r}{2}}(0)$,

we have $y = (y_1 + y) - y_1 = F(G(y_1+y)) - F(G(y_1))$

Denote $x_1 = G(y_1+y)$ and $x_2 = G(y_1)$

Then $y = F(x_1) - F(x_2)$

$$= \left[\int_0^1 DF(x_2 + t(x_1 - x_2))\, dt\right](x_1 - x_2) \qquad \text{(Prop 3.6)}$$

$$= \int_0^1 \left[DF(x_2 + t(x_1-x_2)) - DF(x_2)\right] dt\, (x_1 - x_2) + DF(x_2)(x_1 - x_2)$$

Hence

$$(x_1 - x_2) = (DF)^{-1}(x_2)\, y - (DF)^{-1}(x_2) \int_0^1 \left[DF(x_2 + t(x_1-x_2)) - DF(x_2)\right] dt\, (x_1 - x_2)$$

i.e. $G(y_1+y) - G(y_1) = (DF)^{-1}(G(y_1))\, y + \eta$, $\left(\begin{array}{l} x_1 = G(y_1+y) \\ x_2 = G(y_1) \end{array}\right)$

where $\eta = (DF)^{-1}(x_2) \int_0^1 \left[DF(x_2) - DF(x_2 + t(x_1 - x_2))\right] dt\, (x_1 - x_2)$

Observes that

$$|x_1-x_2| = |G(y_1+y)-G(y_1)| \leq 2\,|(y_1+y)-y_1| = 2|y|, \quad (\text{step2})$$

we have, $|x_1-x_2| \to 0$ as $|y| \to 0$ and

$$\frac{|\eta|}{|y|} \leq 2\|DF^{-1}(x_2)\| \int_0^1 \underbrace{\|DF(x_2)-DF(x_2+t(x_1-x_2))\|}_{\to 0 \text{ as } |x_1-x_2|\to 0}\, dt \quad \left(F \in C^1\right)$$

By assumption F is C^1 ($x_1, x_2 \in \overline{B_r(0)}$), we have

$$\lim_{|y|\to 0} \frac{|\eta|}{|y|} = 0 .$$

Therefore $G(y_1+y)-G(y_1) = (DF)^{-1}(G(y_1))\, y + o(|y|)$,

which implies G is differentiable at $y_1 \in B_{\frac{r}{2}}(0) = W$

and $\quad DG(y_1) = (DF)^{-1}(G(y_1))$. #

Final Step for special case: G is $C^1(B_{\frac{r}{2}}(0))$

Furthermore, if F is C^k ($k \geq 1$), then G is C^k.

Pf of final step:

By assumption, DF is continuous and invertible on $B_r(0)$.

Linear Algebra $\Rightarrow$ $(DF)^{-1}$ is continuous. (eg: using matrix of co-factors)

Therefore, by Step 3 (and step 2)

$$DG(y) = (DF)^{-1}(G(y)) \quad \text{is continuous.}$$

Hence G is C^1.

The fact that $\quad F$ is C^k $(k \geq 1) \Rightarrow G$ is C^k

is by differentiating the identity $DG(y) = (DF)^{-1}(G(y))$

and using induction. ※