3.5 Appendix: Completion of a Metric Space

Def: A metric space (X, d) is said to be isometrically-embedded in metric space (Y, ρ) if

$\exists$ a mapping $\Phi : X \to Y$ s.t.

$$\boxed{d(x,y) = \rho(\Phi(x), \Phi(y))}$$

Notes: (i) Φ is called an isometric embedding from (X, d) into (Y, ρ). And sometime called a metric preserving map.

(ii) Φ must be one-to-one and continuous.

Def: Let (X, d) and (Y, ρ) be metric spaces. We call (Y, ρ) a completion of (X, d) if (1) (Y, ρ) is complete.

(2) $\exists$ isometric embedding $\Phi : (X, d) \to (Y, \rho)$

such that the closure $\boxed{\overline{\Phi(X)} = Y}$.

eg: $(Y,\rho) = (\mathbb{R}, \text{standard})$ is a completion of

$(X,d) = (\mathbb{Q}, \text{induced metric})$ $(X = \mathbb{Q} \subset \mathbb{R})$

Then
- $(\mathbb{R}, \text{standard})$ is complete;
- $$\begin{array}{ccc} \Phi: (\mathbb{Q}, \text{induced metric}) & \to & (\mathbb{R}, \text{standard}) \\ \cup & & \cup \\ q & \longmapsto & q \end{array}$$
- $\overline{\Phi(\mathbb{Q})} = \overline{\mathbb{Q}} = \mathbb{R}$ ($\mathbb{Q}$ is dense in $\mathbb{R}$)

> <u>Def</u>: Two metric spaces (X,d), (X',d') are called <u>isometric</u> if $\exists$ <u>bijective</u> isometric embedding from (X,d) onto (X',d').

<u>Notes</u>: (i) the inverse of the bijective isometric embedding is also an isometric embedding.

(ii) Two metric spaces will be regarded as the <u>same</u> if they are <u>isometric</u>.

> <u>Thm</u>: If (Y,ρ) & (Y',ρ') are completions of a metric space (X,d). Then (Y,ρ) and (Y',ρ') are <u>isometric</u>.
>
> (i.e. Completion is <u>unique up to isometry</u>.)

§3.2 The Contraction Mapping Principle

Def: (1) Let (X, d) be a metric space. A map $T: (X,d) \to (X,d)$ is called a contraction if $\exists$ constant $\gamma \in (0,1)$, such that

$$d(Tx, Ty) \le \gamma\, d(x,y), \quad \forall x, y \in X.$$

(2) A point $x \in X$ is called a fixed point of T if

$$\boxed{Tx = x}$$

(Usually write Tx instead of $T(x)$)

Thm 3.3 (Contraction Mapping Principle) (Banach Fixed Point Thm)

Every contraction in a complete metric space admit a unique fixed point.

Pf: Uniqueness: Suppose x & y are fixed pts. of T.

Then $d(x,y) = d(Tx, Ty)$ (x, y are fixed by T)

$\le \gamma\, d(x,y)$ for some $\gamma \in (0,1)$.

(T contraction)

$\Rightarrow d(x,y) = 0 \Rightarrow x = y$.

Existence: Let $x_0 \in X$.

Define $\{x_n\}_{n=1}^{\infty}$ by $x_n = T x_{n-1}$, for $n=1,2,\dots$

Then $x_n = T x_{n-1} = T(T x_{n-2}) = T^2 x_{n-2}$

$= \dots = T^n x_0$.

For any $n \geq N$,

$$d(x_n, x_N) = d(T^n x_0, T^N x_0) = d(T^{(n-N)+N} x_0, T^N x_0)$$

$$= d(T(T^{(n-N)+N-1} x_0), T(T^{N-1} x_0))$$

$$\leq \gamma\, d(T^{(n-N)+N-1} x_0, T^{N-1} x_0)$$

(where $\gamma \in (0,1)$ is the constant s.t. $d(Tx, Ty) \leq \gamma d(x,y)$, $\forall x,y \in X$)

$$\leq \dots$$

$$\leq \gamma^N d(T^{(n-N)} x_0, x_0) = \gamma^N d(x_0, T^{(n-N)} x_0)$$

$$\leq \gamma^N \Big[d(x_0, Tx_0) + d(Tx_0, T^2 x_0) + \dots + d(T^{(n-N)-2} x_0, T^{(n-N)-1} x_0) + d(T^{(n-N)-1} x_0, T^{(n-N)} x_0) \Big]$$

$$\leq \gamma^N \Big[d(Tx_0, x_0) + \gamma d(Tx_0, x_0) + \dots + \gamma^{(n-N)-2} d(Tx_0, x_0) + \gamma^{(n-N)-1} d(Tx_0, x_0) \Big]$$

$$= \gamma^N \left[1 + \gamma + \cdots + \gamma^{(n-N)-1} \right] d(Tx_0, x_0)$$

$$< \frac{\gamma^N}{1-\gamma} d(Tx_0, x_0)$$

Therefore, $\forall \varepsilon > 0$, if $N > 0$ is chosen s.t.

$$\frac{\gamma^N}{1-\gamma} d(Tx_0, x_0) < \frac{\varepsilon}{2},$$

we have $\forall\, n, m \geq N$,

$$d(x_n, x_m) \leq d(x_n, x_N) + d(x_N, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore$ $\{x_n\}$ is a Cauchy seq. in (X, d).

By completeness of (X, d), $\exists\, x \in X$ s.t. $x_n \to x$.

Note that a contraction is always continuous (Ex!) we have

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = T \lim_{n\to\infty} x_{n-1} = Tx.$$

$\therefore$ x is a fixed point of T. #