Def: Let E be a subset of a metric space (X, d).

(1) A point x is called an <u>interior point</u> of E if $\exists$ an open set G s.t. $x \in G$ & $G \subset E$.

(2) The set of all interior points of E is called the <u>interior of E</u>, denoted by E°.

<u>Notes:</u> (i) E° is open

(ii) $E^\circ = E \setminus \partial E$ (Pf: Ex!)

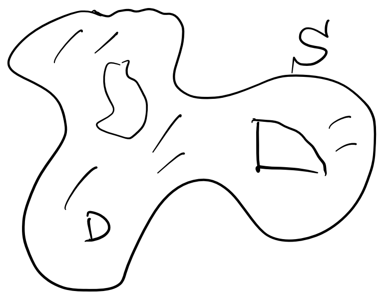
(iii) $E^\circ = X \setminus \overline{(X \setminus E)}$

(iv) $E^\circ = \bigcup \{G : G = \text{open} \ \& \ G \subset E\}$

<u>eg 2.18</u> $E = \mathbb{Q} \cap [0,1]$ in $(X = [0,1], \ d(x,y) = |x-y|)$

Then $E^\circ = \emptyset$ & $\overline{E} = [0,1]$, $\partial E = ?$

<u>eg 2.19</u> Let D be a "domain" in $\mathbb{R}^2$ bounded by several <u>cts.</u> curves S.



Then $\partial D = S$

$\overline{D} = D \cup S = D \cup \partial D$

& $(\overline{D})^\circ = D$.

eg 2.20 :

(i) $\boxed{\overline{E \cup F} = \overline{E} \cup \overline{F}}$ for $E, F \subset (X, d)$ (Ex!)

(ii) However $(E \cup F)^\circ \neq E^\circ \cup F^\circ$ in general.

Counterexample : $(X, d) = (\mathbb{R}, \text{standard})$

$$E = \mathbb{Q} \quad \& \quad F = \mathbb{R} \setminus \mathbb{Q}$$

Then $E \cup F = \mathbb{R} \Rightarrow (E \cup F)^\circ = \mathbb{R}$

However $E^\circ = F^\circ = \emptyset$

$$\Rightarrow \quad E^\circ \cup F^\circ = \emptyset \neq \mathbb{R} = (E \cup F)^\circ.$$

(iii) We only have $\boxed{E^\circ \cup F^\circ \subset (E \cup F)^\circ}$ (Pf: Ex!)

eg 2.21 : $(X = C[0,1],\ d_\infty(f,g) = \|f-g\|_\infty)$

Let $S = \{ f \in X : \alpha < f(x) \leq \beta,\ \forall x \in [0,1] \}$

(1) Claim : $\overline{S} = \{ f \in X : \alpha \leq f(x) \leq \beta,\ \forall x \in [0,1] \}$.

Pf: Let $C = \{ f \in X : \alpha \leq f(x) \leq \beta,\ \forall x \in [0,1] \} \supset S$.

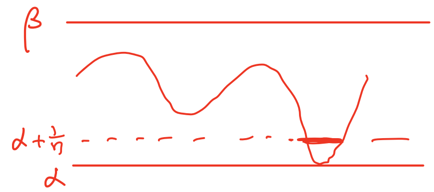
Then $C = \underbrace{\{ \alpha \leq f(x) \} \cap \{ f(x) \leq \beta \}}$ closed in (X, d_∞) (Ex!)

$\therefore C$ is closed. $\Rightarrow \overline{S} \subset C$ (by Prop 2.9 (d))

Conversely, $\forall\, f \in C'$, define

$$f_n(x) = \max\{f(x), \alpha + \tfrac{1}{n}\} \in X = C[0,1], \ \forall n.$$

Then $\alpha < \alpha + \frac{1}{n} \le f_n(x) \le \beta$, $\forall n$ large s.t. $\alpha + \frac{1}{n} < \beta$



$$\Rightarrow \quad f_n \in S', \ \forall n$$

Note $d_\infty(f_n, f) = \max_{x\in[0,1]} |f_n - f|(x)$

$$\le \alpha + \tfrac{1}{n} - \alpha = \tfrac{1}{n} \to 0 \quad \text{as } n \to \infty$$

$\therefore\ f \in \overline{S'}$ as $f_n \in S'$ & $f_n \to f$. Hence $C' \subset \overline{S'}$. #

(2) Claim: $S'^{\,0} = \{f \in X : \alpha < f(x) < \beta, \ \forall x \in [0,1]\}$.

(Pf: Ex!)

§2.5 Elementary Inequalities for Functions

Recall

Young's Inequality

For $a, b > 0$ and $p > 1$,

$$\boxed{ab \leq \frac{a^p}{p} + \frac{b^q}{q}},$$

where q is given by $\boxed{\frac{1}{p} + \frac{1}{q} = 1}$

and "equality holds" $\Leftrightarrow a^p = b^q$.

Note: $q = \frac{p}{p-1} > 1$ is called the conjugate of p.

(Recall Pf: Study the minimum of

$$\varphi(a) = \frac{a^p}{p} + \frac{b^q}{q} - ab. \quad (\text{Ex!}) \text{)}$$

Note: If $p=2$, it is the elementary inequality

$$2ab \leq a^2 + b^2.$$

Thm 2.10 (Hölder's Inequality)

Let $f, g \in R[a,b]$ (Riemann integrable) and $p > 1$.

Then

$$\boxed{\int_a^b |f(x)g(x)|\,dx \le \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}}}$$

where $q = \frac{p}{p-1}$ is the conjugate of p.

"Equality holds" $\Leftrightarrow$

either (a) f or $g = 0$ almost everywhere,

or (b) $\exists$ constant $\lambda > 0$ s.t.

$$|g(x)|^q = \lambda |f(x)|^p \text{ almost everywhere.}$$

($\Leftrightarrow$ $\exists$ constants $\lambda_1, \lambda_2 \ge 0$, not both zero, such that

$$\lambda_1 |f(x)|^p = \lambda_2 |g(x)|^q \text{ a.e.})$$

Note: If we denote $\boxed{\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}}$

Then the Hölder Inequality can be written as

$$\boxed{\int_a^b |f(x)g(x)|\,dx \le \|f\|_p \|g\|_q}$$

Pf: If $\|f\|_p = 0$, or $\|g\|_q = 0$.

Then $f = 0$ or $g = 0$ almost everywhere.

Hence $0 = \int_a^b |f(x)g(x)|dx$

and the inequality holds trivially.

Assume now that $\|f\|_p > 0$ and $\|g\|_q > 0$.

By Young's Inequality, we have for any $\varepsilon > 0$,

$$|f(x)g(x)| = \left|\varepsilon f(x) \cdot \frac{g(x)}{\varepsilon}\right|$$

$$\leq \frac{\varepsilon^p |f(x)|^p}{p} + \frac{|g(x)|^q}{q\varepsilon^q}, \qquad \forall x \in [a,b]$$

$\Rightarrow$

$$\int_a^b |f(x)g(x)|dx \leq \frac{\varepsilon^p}{p}\int_a^b |f(x)|^p dx + \frac{1}{q\varepsilon^q}\int_a^b |g(x)|^q dx$$

$$= \frac{\varepsilon^p}{p}\|f\|_p^p + \frac{1}{q\varepsilon^q}\|g\|_q^q$$

Choose $\varepsilon > 0$ s.t. $\varepsilon^p \|f\|_p^p = \frac{1}{\varepsilon^q}\|g\|_q^q$

i.e. $$\varepsilon^{p+q} = \frac{\|g\|_q^q}{\|f\|_p^p} \quad \left(= \frac{\int_a^b |g(x)|^q dx}{\int_a^b |f(x)|^p dx} \right)$$

$$\Rightarrow \quad \varepsilon = \frac{\|g\|_q^{\frac{q}{p+q}}}{\|f\|_p^{\frac{p}{p+q}}} > 0 .$$

Then

$$\int_a^b |f(x)g(x)|dx \leq \frac{\varepsilon^p}{p}\|f\|_p^p + \frac{1}{q\varepsilon^q}\|g\|_q^q$$

$$= \left(\frac{1}{p}+\frac{1}{q}\right)\varepsilon^p \|f\|_p^p$$

$$= \frac{\|g\|_q^{\frac{pq}{p+q}}}{\|f\|_p^{\frac{p^2}{p+q}}} \cdot \|f\|_p^p \qquad \left(\text{using } \frac{1}{p}+\frac{1}{q}=1 \Updownarrow \frac{p+q}{pq}=1 \right)$$

$$= \|g\|_q \|f\|_p^{p\left(1-\frac{p}{p+q}\right)}$$

$$= \|g\|_q \|f\|_p^{p\cdot\frac{q}{p+q}}$$

$$= \|g\|_q \|f\|_p .$$

From the proof, "Equality holds"

$$\Leftrightarrow \quad |f(x)g(x)| = \frac{\varepsilon^p |f(x)|^p}{p} + \frac{|g(x)|^q}{q\,\varepsilon^q} \quad \text{almost everywhere}$$

(with ε given above)

$$\Leftrightarrow \quad \varepsilon^p |f(x)|^p = \frac{|g(x)|^q}{\varepsilon^q} \quad \text{almost everywhere}$$

$$\Leftrightarrow \quad |g(x)|^q = \varepsilon^{p+q} |f(x)|^q \quad \text{almost everywhere.}$$

$$\Rightarrow \quad \exists\, \lambda = \varepsilon^{p+q} > 0 \text{ s.t. } |g(x)|^q = \lambda |f(x)|^p \text{ a.e.}$$

Conversely, if $|g(x)|^q = \lambda |f(x)|^p$ a.e. for some $\lambda > 0$, we clearly have the "Equality". ⋇

<u>Note:</u> <u>Limiting cases</u>

(Note: Riemann integrable $\Rightarrow$ bounded)

(i) $p \to 1$ ($\Rightarrow q \to +\infty$)

$$\boxed{\int_a^b |f(x)g(x)|\,dx \le \|f\|_1 \|g\|_\infty}$$

(ii) $p \to +\infty$ ($\Rightarrow q \to 1$)

$$\boxed{\int_a^b |f(x)g(x)|\,dx \le \|f\|_\infty \|g\|_1}$$

Thm 2.11 (Minkowski's Inequality)

$\forall f, g \in R[a,b]$, and $p > 1$,

$$\boxed{\|f+g\|_p \le \|f\|_p + \|g\|_p}$$

"Equality holds" $\Leftrightarrow$

either (a) f or $g = 0$ a.e.

or (b) $\|f\|_p > 0$, $\|g\|_p > 0$ and $\exists$ constant

$\lambda > 0$ s.t. $g(x) = \lambda f(x)$ a.e.

($\Leftrightarrow$ $\exists$ constants $\lambda_1, \lambda_2 \ge 0$, not both zero, s.t.

$\lambda_1 f(x) = \lambda_2 g(x)$ a.e.)

Note: Minkowski's Inequality follows from the Hölder's Inequality. However, we will provide an alternate proof without using Hölder's Inequality which is simpler to prove the "equality" case.

Lemma: $\varphi(t) = t^p$, $p > 1$, is strictly convex on $[0,\infty)$.

(in fact, $\varphi''(t) > 0$, $\forall\, t \in (0,\infty)$)

Hence $\forall\, a, b > 0$ and $\lambda \in [0,1]$,

$$\boxed{[(1-\lambda)a + \lambda b]^p \leq (1-\lambda)a^p + \lambda b^p}$$

"Equality holds" $\Leftrightarrow$ $a = b$

Pf: (Sketch)

Consider $\psi(\lambda) = [(1-\lambda)a + \lambda b]^p - (1-\lambda)a^p - \lambda b^p$, for $\lambda \in [0,1]$

Use $\varphi''(t) > 0$ to show that

$\psi''(\lambda) \geq 0$ for $\lambda \in (0,1]$ ("equality" $\Leftrightarrow$ $b = a$)

Hence maximum of ψ attained at the boundary

$\lambda = 0$ or $\lambda = 1$.

Since $\psi(0) = 0 = \psi(1)$,

$\psi(\lambda) \leq 0$, $\forall\, \lambda \in [0,1]$.

This also shows the "equality" case. $\#$

Pf of Minkowski's Inequality

If $\|f\|_p = 0$, then $f(x) = 0$ a.e. $x \in [a,b]$,

the inequality in fact an equality.

Similarly if $\|g\|_p = 0$.

Assume $\|f\|_p > 0$ and $\|g\|_p > 0$.

Then

$$\int_a^b |f+g|^p(x)\,dx$$

$$\le \int_a^b (|f|+|g|)^p(x)\,dx \quad (\text{since } p>1)$$

$$= (\|f\|_p + \|g\|_p)^p \int_a^b \left[\frac{\|f\|_p}{\|f\|_p+\|g\|_p} \cdot \frac{|f(x)|}{\|f\|_p} + \frac{\|g\|_p}{\|f\|_p+\|g\|_p} \cdot \frac{|g(x)|}{\|g\|_p} \right]^p dx$$

Since $\dfrac{\|f\|_p}{\|f\|_p+\|g\|_p} + \dfrac{\|g\|_p}{\|f\|_p+\|g\|_p} = 1$,

the lemma $\Rightarrow$

$$\int_a^b |f+g|^p(x)\,dx$$

$$\le (\|f\|_p + \|g\|_p)^p \int_a^b \left[\frac{\|f\|_p}{\|f\|_p+\|g\|_p} \cdot \left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{\|g\|_p}{\|f\|_p+\|g\|_p} \cdot \left(\frac{|g(x)|}{\|g\|_p}\right)^p \right] dx$$

$$= (\|f\|_p + \|g\|_p)^p \left[\frac{\|f\|_p}{\|f\|_p + \|g\|_p} \cdot \frac{\int_a^b |f(x)|^p dx}{\|f\|_p^p} + \frac{\|g\|_p}{\|f\|_p + \|g\|_p} \frac{\int_a^b |g(x)|^p dx}{\|g\|_p^p} \right]$$

$$= (\|f\|_p + \|g\|_p)^p \left[\frac{\|f\|_p}{\|f\|_p + \|g\|_p} + \frac{\|g\|_p}{\|f\|_p + \|g\|_p} \right]$$

$$= (\|f\|_p + \|g\|_p)^p$$

$$\therefore \quad \|f+g\|_p \leq \|f\|_p + \|g\|_p .$$

Equality holds : (if $\|f\|_p > 0$, $\|g\|_p > 0$)

$\Leftrightarrow$ (1) $|f+g| = |f| + |g|$ a.e. and

(2) $\dfrac{|f(x)|}{\|f\|_p} = \dfrac{|g(x)|}{\|g\|_p}$ a.e. (By Lemma)

It is clear now from (1) & (2)

$$f(x) = \lambda g(x) \text{ a.e. } x \in [a,b]$$

for the positive constant $\lambda = \dfrac{\|f\|_p}{\|g\|_p} > 0$. #

__Remark__ : Minkowski's inequality $\Rightarrow$ $\|f\|_p$ for $p > 1$ is a __norm__ on $R[a,b]/\sim$ ($\leftarrow$ relation mod a.e.)

Ch3 The Contraction Mapping Principle

§3.1 Complete Metric Space

Def: Let (X, d) be a metric space.

(1) A sequence $\{x_n\}$ in (X, d) is a Cauchy sequence if $\forall \varepsilon > 0$, $\exists n_0$ s.t. $d(x_n, x_m) < \varepsilon$, $\forall n, m \geq n_0$.

(2) (X, d) is complete if every Cauchy sequence in (X, d) converges.

(3) A subset E is complete if the induce metric subspace (E, d) is complete. (i.e. $d = d|_{E \times E}$)

(i.e. every Cauchy sequence in E converges with limit in E.)

Note: Convergent sequence is a Cauchy sequence (Ex!)

Prop 3.1 Let (X, d) be a metric space.

(a) Every complete set in X is closed.

(b) If X is complete, then every closed set in X is complete.

Pf: (a) Let $E \subset X$, & E complete.

Suppose $\{x_n\} \subset E$ with $x_n \to x$ in X.

By note, $\{x_n\}$ is a Cauchy seq. in E

Then completeness of $E \Rightarrow x_n \to z \in E$.

Uniqueness of limit $\Rightarrow x = z \in E$

$\therefore$ E is closed.

(b) Let (X, d) be complete & E is closed in X.

Then every Cauchy seq. $\{x_n\}$ in E is a Cauchy seq in X.

Completeness of $X \Rightarrow \exists x \in X$, s.t. $x_n \to x$.

Since E is closed, $x \in E$.

$\therefore$ E is complete. ※

eg 3.1: • $(\mathbb{R}$, standard) is complete

• $[a,b]$, $(-\infty, b]$, $[a, \infty)$ complete

• $[a,b)$ (b finite) not complete ($\because x_n = b - \frac{1}{n} \to b \notin [a,b)$)

• $\mathbb{Q}$ is not complete.

eg 3.2 $(X = C[a,b], d_\infty)$ is complete :

Cauchy seq $\{f_n\}$ in d_∞-metric

$\Leftrightarrow \forall \varepsilon > 0, \exists n_0$ s.t.

$$\max_{[a,b]} |f_n(x) - f_m(x)| < \varepsilon, \quad \forall n, m \geq n_0$$

$\therefore f_n(x) \to f(x)$ uniformly for some $f \in C[a,b]$

※

eg Let $P = \left\{ f \in C[a,b] : f(x) = p(x) \text{ on } [a,b] \text{ for some polynomial } p(x) \right\}$

Then P is <u>not</u> complete (in d_∞-metric):

$$h_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \in P$$

but $h_n(x) \to e^x$ uniformly (in d_∞-metric)

& $e^x \notin P$.