§2.4 Points in Metric Spaces

Def: Let E be a set in a metric space (X, d)

(1) A point $x \in X$ (not neccessary in E) is called a <u>boundary point</u> of E if

$\forall$ open set $G \subset X$ containing x,

$$G \cap E \neq \phi \quad \& \quad G \setminus E \neq \phi$$

$$(G \cap (X \setminus E) \neq \phi)$$

(2) The set of boundary points of E will be denoted by ∂E and is called the <u>boundary of E</u>.

(3) The <u>closure</u> of E, denoted by $\overline{E}$, is defined to be

$$\overline{E} = E \cup \partial E.$$

<u>Note</u>: (i) In (1), it suffices to check G of the form $B_\varepsilon(x)$ for all small $\varepsilon > 0$, or even just $B_{\frac{1}{n}}(x)$, $\forall n \geq 1$ (See the proof of Prop 2.9(a)).

(ii) $\partial E = \partial(X \setminus E)$, $\forall E \subset X$.

$$E \left\{ \begin{array}{l} X \setminus E \\ \partial E \end{array} \right.$$

eg: For $B_r(x) = \{y \in X : d(y,x) < r\}$ in $(\mathbb{R}^n, \text{standard})$

$$\partial B_r(x) = S_r(x) = \{y \in X : d(y,x) = r\} \quad \&$$

$$\overline{B_r}(x) = B_r(x) \cup \partial B_r(x) = \{y \in X : d(y,x) \le r\}$$

<u>Further Notes</u> (i) $\partial \emptyset = \emptyset$ (Ex!)

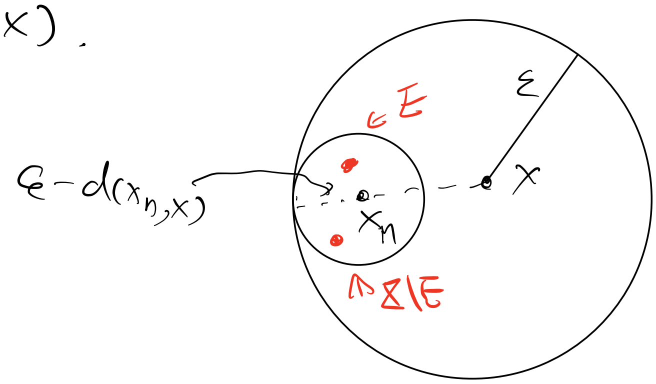
(ii) $\forall\, E \subset X$, ∂E is a closed set.

(iii) If E is closed, then $\overline{E} = E$.

<u>Pf of (ii)</u>: Consider a seq $\{x_n\} \subset \partial E$ converging to some $x \in X$.

Then $\forall\, \varepsilon > 0$, $x_n \in B_\varepsilon(x)$ for $n \ge n_0$ (for some n_0)

$$\Rightarrow \quad B_{\varepsilon - d(x_n,x)}(x_n) \subset B_\varepsilon(x).$$



As $x_n \in \partial E$,

$$\begin{cases} B_{\varepsilon - d(x_n,x)}(x_n) \cap E \neq \emptyset \\ B_{\varepsilon - d(x_n,x)}(x_n) \setminus E \neq \emptyset \end{cases}$$

$$\Rightarrow \begin{cases} B_\varepsilon(x) \cap E \neq \emptyset \\ B_\varepsilon(x) \setminus E \neq \emptyset \end{cases} \underset{\left(\text{since } \varepsilon>0 \text{ arbitrary}\right)}{\Longrightarrow} x \in \partial E.$$

Therefore ∂E is closed. ⋇

Pf of (iii): Only need to show that

$$\partial E \subset E \quad \text{if } E \text{ is closed.}$$

Let $x \in \partial E$, then by definition

$$B_{\frac{1}{n}}(x) \cap E \neq \emptyset \quad (\& \ B_{\frac{1}{n}}(x) \cap (X \setminus E) \neq \emptyset)$$

$$\Rightarrow \quad \exists \ x_n \in B_{\frac{1}{n}}(x) \cap E.$$

$$\Rightarrow \quad d(x_n, x) < \frac{1}{n}, \quad \forall n$$

$$\therefore \ x_n \to x$$

Since E is closed, Prop 2.7 $\Rightarrow$ $x \in E$.

Since $x \in \partial E$ is arbitrary, $\partial E \subset E$. #

<u>Prop 2.9</u> Let $E \subset (X, d)$. Then

(a) $x \in \overline{E} \iff B_r(x) \cap E \neq \emptyset, \ \forall r > 0$.

(b) $A \subset B \Rightarrow \overline{A} \subset \overline{B} \quad \forall A, B \subset (X, d)$

(c) $\overline{E}$ is closed

(d) $\overline{E} = \cap \{C : C = \text{closed set}, C \supset E\}$.

(i.e. $\overline{E}$ is the smallest closed set containing E)

Pf (a) ($\Rightarrow$)

$x \in \overline{E} \Rightarrow x \in E$ or $x \in \partial E$.

If $x \in E$, then $x \in B_r(x) \cap E$, $\forall r > 0$

$$\Rightarrow B_r(x) \cap E \neq \phi, \forall r > 0.$$

If $x \in \partial E$, then by definition of boundary point,

$\forall$ open set G containing x, $G \cap E \neq \phi$ ($\&$ $G \setminus E \neq \phi$)

Since $B_r(x)$ is open and $x \in B_r(x)$, $\forall r > 0$,

we have $B_r(x) \cap E \neq \phi$, $\forall r > 0$.

($\Leftarrow$)

If $x \in E$, we are done. ($x \in E \subset \overline{E}$)

If $x \notin E$, then for any open set G containing x,

$$x \in G \setminus E.$$

Hence $G \setminus E \neq \phi$.

To show that $G \cap E \neq \phi$, we choose $r_0 > 0$

s.t. $B_{r_0}(x) \subset G$ (it is possible since G is open)

Then by assumption, $B_{r_0}(x) \cap E \neq \phi$

and hence $G \cap E \left(\supset B_{r_0}(x) \cap E\right) \neq \phi$.

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(b) Let $x \in \overline{A}$.

By part (a), $B_r(x) \cap A \neq \emptyset$, $\forall r > 0$

Since $A \subset B$, $B_r(x) \cap B \neq \emptyset$, $\forall r > 0$

Part (a) again, $x \in \overline{B}$.

$\therefore \overline{A} \subset \overline{B}$. #

(c) Consider a seq $\{x_n\} \in \overline{E}$ such that $x_n \to x$ for some $x \in X$. We need to show that $x \in \overline{E}$. (Prop 2.7)

Suppose not, then $x \notin \overline{E}$.

Part (a) $\Rightarrow$ $\exists\ \varepsilon_0 > 0$ such that $B_{\varepsilon_0}(x) \cap E = \emptyset$

For this $\varepsilon_0 > 0$, $\exists\ n_0 > 0$ such that

$$x_n \in B_{\varepsilon_0}(x) \quad \forall n \geq n_0.$$

Then $B_{\varepsilon_0}(x) \cap E = \emptyset \Rightarrow x_n \in \partial E \setminus E$ for $n \geq n_0$.

In particular $\{x_n\}_{n=n_0}^{\infty}$ is a seq. in ∂E and $x_n \to x$.

By Note (ii) above and prop 2.7, $x \in \partial E \subset \overline{E}$

which is a contradiction. #

(d) By (c), $\bar{E}$ is closed & $\bar{E} \supset E$

$$\therefore \quad \bar{E} \in \{ C : C = \text{closed set}, C \supset E \}$$

$$\Rightarrow \quad \bar{E} \supset \bigcap \{ C : C = \text{closed set}, C \supset E \}$$

Conversely, let C be a closed set & $C \supset E$.

Then by (b) and (iii) of Further Notes above,

$$\bar{E} \subset \bar{C} = C$$

$$\Rightarrow \quad \bar{E} \subset \bigcap \{ C : C = \text{closed set}, C \supset E \}. \quad \#$$