Eg: Let (X, d) be a metric space, $A \subset X$, $A \neq \phi$.

Define $d(x, A) : X \to \mathbb{R}$ by

$$\boxed{d(x,A) = \inf_{y \in A} d(x,y)}$$

(distance from x to the subset A)

$\boxed{\text{Claim: } |d(x,A) - d(y,A)| \leq d(x,y), \ \forall x,y \in X.}$

Pf of claim For fixed $x, y \in X$.

By defn. of $d(y,A)$,

$\forall \varepsilon > 0$, $\exists z \in A$ s.t. $d(y,A) + \varepsilon > d(z,y)$

Hence,

$$\begin{aligned} d(x,A) &\leq d(z,x) \\ &\leq d(x,y) + d(y,z) \quad \text{(triangle ineq.)} \\ &< d(x,y) + d(y,A) + \varepsilon \end{aligned}$$

$$\Rightarrow \quad d(x,A) - d(y,A) < d(x,y) + \varepsilon$$

Interchanging the roles of x & y, we have

$$d(y,A) - d(x,A) < d(y,x) + \varepsilon = d(x,y) + \varepsilon$$

Therefore $|d(x,A) - d(y,A)| < d(x,y) + \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $|d(x,A) - d(y,A)| \leq d(x,y)$.

#

<u>Note</u>: By claim, $d(x_n, x) \to 0 \Rightarrow d(x_n, A) \to d(x, A)$

$\therefore$ $d(x, A) : (X, d) \to \mathbb{R}$ is cts (as a function of x)

(In fact, $d(x, A)$ is "Lipschitz continuous")

This example shows that <u>there are "many" cts functions</u> on a metric space.

<u>Notation</u>: Usually, we use the following notations for subsets A & $B \subset X$

$$\begin{cases} d(x, A) = \inf \{ d(x,y) : y \in A \} \\ d(A, B) = \inf \{ d(x,y) : x \in A, y \in B \} \end{cases}$$

§2.3 Open and Closed Sets

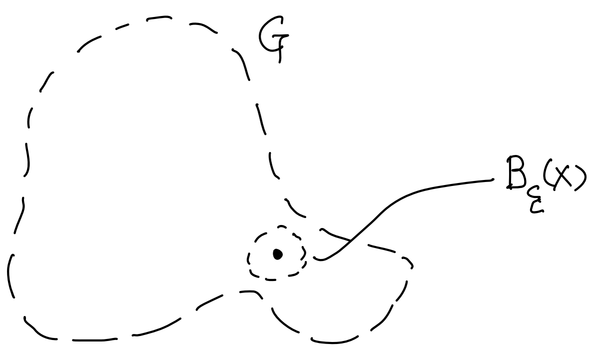
Def: Let (X, d) = metric space

- A set $G \subset X$ is called an open set if

  $\forall x \in G$, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) = \{y : d(y,x) < \varepsilon\} \subset G$.

  (The number $\varepsilon > 0$ may vary depending on x)

- We also define the empty set $\emptyset$ to be an open set.



Prop 2.4: Let (X, d) be a metric space. We have

(a) X and $\emptyset$ are open sets.

(b) Arbitrary union of open sets is open : if G_α, $\alpha \in \mathcal{A}$, is a collection of open sets, then $\bigcup_{\alpha \in \mathcal{A}} G_\alpha$ is an open set.

(c) Finite intersection of open sets is open : If $G_1, \cdots, G_N$ are open sets, then $\bigcap_{j=1}^{N} G_j$ is an open set.

Pf: (a) Clear

(b) Let $x \in \bigcup_{\alpha \in \mathcal{A}} G_\alpha$

$\Rightarrow x \in G_\alpha$ for some $\alpha \in \mathcal{A}$

$\Rightarrow \exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset G_\alpha$ (Since G_α open)

$\Rightarrow B_\varepsilon(x) \subset \bigcup_{\alpha \in \mathcal{A}} G_\alpha$

(c) Let $x \in \bigcap_{j=1}^{N} G_j \Rightarrow x \in G_j, \ \forall j = 1, \cdots, N$

$\Rightarrow \exists\, \varepsilon_j > 0$ s.t. $B_{\varepsilon_j}(x) \subset G_j, \ \forall j = 1, \cdots, N.$

Let $\varepsilon = \min\{\varepsilon_1, \cdots, \varepsilon_N\} > 0$. Then

$B_\varepsilon(x) \subset B_{\varepsilon_j}(x) \subset G_j, \ \forall j = 1, \cdots, N$

$\Rightarrow B_\varepsilon(x) \subset \bigcap_{j=1}^{N} G_j$. ※

Def: Let (X, d) be a metric space.

A set $F \subset X$ is called a closed set if the complement $X \setminus F$ is an open set.

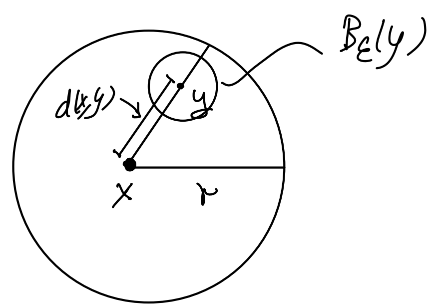
$\boxed{\begin{array}{l}\underline{\text{Prop 2.5}}: \text{ Let } (X, d) \text{ be a metric space. We have} \\ \text{(a) } X \text{ and } \emptyset \text{ are closed sets.} \\ \text{(b) } \underline{\text{Arbitrary intersection of closed sets is closed}}: \text{ if } F_\alpha, \alpha \in \mathcal{A}, \\ \quad \text{is a collection of closed sets, then } \bigcap_{\alpha \in \mathcal{A}} F_\alpha \text{ is a closed set.} \\ \text{(c) } \underline{\text{Finite union of closed sets is closed}}: \text{ If } F_1, \cdots, F_N \text{ are closed sets,} \\ \quad \text{then } \bigcup_{j=1}^{N} F_j \text{ is a closed set.}\end{array}}$

<u>Note:</u> Prop 2.4 & 2.5 $\Rightarrow$ X & $\emptyset$ are <u>both open & closed</u>.

<u>eg 2.10</u> (1) Every <u>metric ball</u>

$$\boxed{B_r(x) = \{y \in X : d(x,y) < r\} \quad (r > 0)}$$

is an open set.



<u>Pf</u>: $\forall\, y \in B_r(x)$

Then $\varepsilon \overset{\text{def}}{=} r - d(x,y) > 0$

& $\forall\, z \in B_\varepsilon(y)$

$$d(z,x) \leq d(z,y) + d(y,x) < \varepsilon + d(y,x) = r$$

$$\Rightarrow \quad B_\varepsilon(y) \subset B_r(x)$$

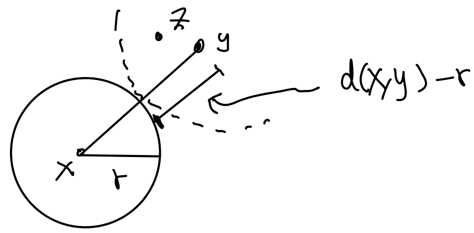
$\divideontimes$

(2) The set $E = \{y \in X : d(y,x) > r\}$ (for a fixed $x \in X$)

is open and hence

$$X \setminus E = \{y \in X : d(y,x) \le r\} \text{ is closed.}$$

Pf: $\forall\, y \in E$



Then $\varepsilon \overset{\text{def}}{=} d(x,y) - r > 0$

$\forall\, z \in B_\varepsilon(y)$

$$d(z,x) \ge d(x,y) - d(z,y) > d(x,y) - (d(x,y) - r) = r$$

$\therefore\ B_\varepsilon(y) \subset E$ ※

<u>Note</u>: We usually write

$$\boxed{\overline{B_r}(x) = \overline{B_r(x)} = \{y \in X : d(y,x) \le r\}}$$

the <u>closed ball</u> of radius r centered at x.

(Confusing notation here, may not equal to the "closure" of $B_r(x)$ in a general metric space.)

(3) Since $B_r(x)$ & $E = \{y \in X : d(x,y) > r\}$ are open,

$B_r(x) \cup E$ is open

$\Rightarrow \quad X \setminus (B_r(x) \cup E) = \{y \in X : d(x,y) = r\}$ is closed.

In particular, $E = \{y \in X : d(x,y) > 0\}$ is open (take $r = 0$)

$\Rightarrow \{x\} = X \setminus E$ is <u>closed</u>. (in any metric space)

(Note: $\{x\}$ may <u>not</u> be open (unless $\exists\, \varepsilon_0 > 0$ s.t. $B_{\varepsilon_0}(x) = \{x\}$))

<u>eg2.11</u> $B_{\frac{1}{n}}(x)$, $n = 1, 2, \cdots$ are open sets

<u>Claim</u> $\bigcap_{n=1}^{\infty} B_{\frac{1}{n}}(x) = \{x\}$ (closed, may not be open)

(this shows countable infinite intersection of open sets may not be open)

<u>Pf of claim</u>: $\forall\, y \in \bigcap_{n=1}^{\infty} B_{\frac{1}{n}}(x) \Rightarrow y \in B_{\frac{1}{n}}(x)$, $\forall\, n = 1, 2, \cdots$

$\Rightarrow d(y,x) < \frac{1}{n}$, $\forall n$

$\Rightarrow d(y,x) = 0$

$\Rightarrow y = x$

$\#$

eg 2.13 $X = C[a,b]$ with $d_\infty(f,g) = \|f-g\|_\infty = \sup_{x\in[a,b]} |f-g|(x)$

let

$$E = \{ f \in C[a,b] : f(x) > 0, \ \forall x \in [a,b] \} \subset X$$

$\forall f \in E$, f is positive, cts on the closed & bounded interval $[a,b]$, therefore $\exists\, m > 0$ s.t.

$$f(x) \geq m > 0, \qquad \forall x \in [a,b].$$

Consider

$$B^\infty_{\frac{m}{2}}(f) = \{ g \in C[a,b] : d_\infty(g,f) < \tfrac{m}{2} \}$$

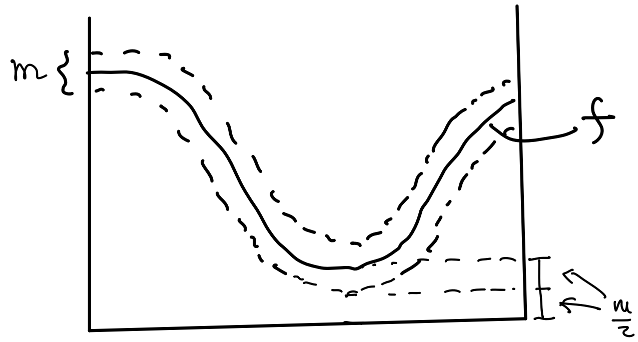
Then

$\forall g \in B^\infty_{\frac{m}{2}}(f)$, we have $\forall x \in [a,b]$.

$$g(x) = [g(x) - f(x)] + f(x)$$

$$\geq f(x) - \|g - f\|_\infty$$

$$> f(x) - \frac{m}{2} \geq m - \frac{m}{2} = \frac{m}{2} > 0$$



$\therefore$ $g \in E$ & hence $B^{\infty}_{\frac{m}{2}}(f) \subset E$

$\therefore$ E is open in $(C[a,b], d_\infty)$

$\left(\text{as } \forall f \in E, \quad \exists\, B^{\infty}_{\frac{m}{2}}(f) \subset E \right)$

Similarly, one can show that $\forall \alpha \in \mathbb{R}$

$$\{ f \in C[a,b] : f(x) > \alpha, \ \forall x \in [a,b] \}$$

$$\{ f \in C[a,b] : f(x) < \alpha, \ \forall x \in [a,b] \}$$

are open in $(C[a,b], d_\infty)$.

And $\{ f \in C[a,b] : f(x) \geq \alpha, \ \forall x \in [a,b] \}$

$\{ f \in C[a,b] : f(x) \leq \alpha, \ \forall x \in [a,b] \}$

are closed in $(C[a,b], d_\infty)$ (Ex!)

$$\left(\begin{array}{l} \text{Caution: } C[a,b] \setminus \{ f \in C[a,b] : f(x) \geq \alpha, \ \forall x \in [a,b] \} \\ \qquad \neq \{ f \in C[a,b] : f(x) < \alpha, \ \forall x \in [a,b] \} \end{array} \right)$$

eg 2.14: Let $X \neq \emptyset$ and d = discrete metric on X.

Then $\forall$ subset $E \subset X$,

$$B_{\frac{1}{2}}(x) = \{x\} \subset E, \quad \forall x \in E.$$

$\therefore$ E is open.

Therefore, any subset E of $(X, \text{discrete})$ is open,

& hence any subset E of $(X, \text{discrete})$ is closed.

Together, any subset E of $(X, \text{discrete})$ is

<u>both open and closed</u>.

In particular, any $\{x\} \subset (X, \text{discrete})$ is

<u>both open and closed</u>.

Prop 2.6 Let (X, d) be a metric space.

A sequence $\{x_n\}$ converges to x <u>if and only if</u>

$\forall$ open set G containing x, $\exists\, n_0$ such that

$$x_n \in G, \quad \forall n \geq n_0.$$

Pf: ($\Rightarrow$) Let G open & $x \in G$.

$\Rightarrow \exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset G$

As $x_n \to x$, for this $\varepsilon > 0$, $\exists\, n_0$ s.t.

$$d(x_n, x) < \varepsilon, \quad \forall\, n \geq n_0$$

$$\Rightarrow \quad x_n \in B_\varepsilon(x) \subset G, \quad \forall\, n \geq n_0$$

($\Leftarrow$) $\forall\, \varepsilon > 0$, $B_\varepsilon(x)$ is an open set containing x.

Therefore $\exists\, n_0$ s.t. $x_n \in B_\varepsilon(x)$, $\forall\, n \geq n_0$

$$\Rightarrow \quad d(x_n, x) < \varepsilon, \ \forall\, n \geq n_0 \quad \#$$

Prop 2.7 Let (X, d) be a metric space.

Then a set $A \subset X$ is closed

if and only if

whenever $\{x_n\} \subset A$ and $x_n \to x$ as $n \to \infty$

implies that $x \in A$.

Pf: ($\Rightarrow$) Suppose not. Then $x \notin A$

i.e. $x \in X \setminus A$ which is open (as A closed)

$\Rightarrow \exists \varepsilon > 0,\ B_\varepsilon(x) \subset X \setminus A$.

On the other hand $x_n \to x$,

$\Rightarrow \exists n_0$ s.t. $d(x_n, x) < \varepsilon \quad \forall n \geq n_0$

$\Rightarrow x_n \in B_\varepsilon(x) \subset X \setminus A$ (by above)

$\Rightarrow x_n \notin A$ contradiction ※

($\Leftarrow$) Suppose <u>not</u>. Then A is not closed.

$\Leftrightarrow X \setminus A$ is not open

$\exists x \in X \setminus A$ s.t. $B_\varepsilon(x) \not\subset X \setminus A,\ \forall \varepsilon > 0$.

In particular, $B_{\frac{1}{n}}(x) \cap A \neq \emptyset,\ \forall n = 1, 2, \ldots$

Pick $x_n \in B_{\frac{1}{n}}(x) \cap A$ for each n

Then $\{x_n\} \subset A$ & $d(x_n, x) < \frac{1}{n},\ \forall n$

$\Rightarrow x_n \to x$ as $n \to \infty$.

Contradicting the assumption (as $x \in X \setminus A$.) ※

Prop 2.8 Let $f:(X,d)\to(Y,\rho)$ be a mapping between metric spaces.

(a) f is continuous at x

$\iff$ $\forall$ open set G (in Y) containing $f(x)$,

$f^{-1}(G)$ contains $B_\varepsilon(x)$ for some $\varepsilon>0$.

(b) f is continuous in X

$\iff$ $\forall$ open set G in Y, $f^{-1}(G)$ is open in X

Pf: (a) ($\Rightarrow$) Suppose not,

then $\exists$ open set G in Y containing $f(x)$

s.t. $f^{-1}(G)$ doesn't containing $B_\varepsilon(x)$, $\forall \varepsilon>0$.

i.e. $B_\varepsilon(x)\cap[X\setminus f^{-1}(G)]\neq\phi$, $\forall\varepsilon>0$.

In particular $B_{\frac{1}{n}}(x)\cap[X\setminus f^{-1}(G)]\neq\phi$, $\forall n$.

Pick $x_n\in B_{\frac{1}{n}}(x)\cap[X\setminus f^{-1}(G)]$, $\forall n$.

Then

$$\begin{cases} x_n\in B_{\frac{1}{n}}(x) \Rightarrow x_n\to x \text{ as } n\to\infty \\ x_n\in X\setminus f^{-1}(G)\Rightarrow f(x_n)\notin G,\ \forall n \end{cases}$$

By Prop 2.6, $f(x_n) \not\to f(x)$. Contradicting the assumption that f is cts. at x.

$(\Leftarrow)$ $\forall \varepsilon > 0$, $B_\varepsilon(f(x)) \subset Y$ is an open set containing $f(x)$. By assumption,

$$f^{-1}\left(B_\varepsilon(f(x))\right) \supset B_\delta(x) \text{ for some } \delta > 0$$

i.e $\quad f(y) \in B_\varepsilon(f(x)), \quad \forall\, y \in B_\delta(x)$

$\Rightarrow \quad d(f(y), f(x)) < \varepsilon, \quad \forall\, d(y, x) < \delta.$

$\therefore$ f is cts. at x.

(b) follows from (a). (Ex!) $\#$

<u>Note</u>: We also have:

f is cts in X $\iff$
$\forall$ closed set $F \subset Y$, $f^{-1}(F)$ is closed in X.

(Pf: Ex!)

Eg: (i) Let $A \subset X$ & $A \neq \emptyset$.

Since $f(x) = d(x, A)$ is cts

$$G_r = \{x \in X : d(x, A) < r\} = f^{-1}(B_r(0))$$ in $\mathbb{R}$

is open in X.

(ii) Claim: If $A (\neq \emptyset)$ is closed,

then $A = \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$. $\left(G_{\frac{1}{n}} = \{x \in X : d(x, A) < \frac{1}{n}\}\right)$

Hence any closed set is a countable intersection of open sets.

Pf: It is clear that $A \subset \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$ as $A \subset G_{\frac{1}{n}}, \forall n$.

Let $x \in \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$, then $x \in G_{\frac{1}{n}}$, $\forall n$

$\Rightarrow$ $d(x, A) < \frac{1}{n}$, $\forall n$

$\Rightarrow$ $\exists\, x_n \in A$ s.t. $d(x, x_n) < \frac{1}{n}$, $\forall n$

Hence $\{x_n\} \subset A$ is a seq in A s.t. $x_n \to x$.

Since A is closed, we have $x \in A$. (Prop 2.7)

$\therefore$ $A = \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$ $\#$