## §2.2 Limits & Continuity

**Def**: A sequence $\{x_n\}$ in a metric space $(X, d)$ is said to be converge to $x \in X$ if

$$\boxed{\lim_{n\to\infty} d(x_n, x) = 0}$$

In this case, we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ in $X$.

**Prop** (Uniqueness of limit)

If $x_n \to x$ & $x_n \to y$ in a metric space, then $x = y$

(Pf: same as in $\mathbb{R}^n$ by using (M1))

egs (i) Convergence in $(\mathbb{R}^n, d_2)$ is the usual convergence in Adv. Calculus.

(ii) Convergence in $(C[a,b], d_\infty)$ is the uniform convergence of a seq. of functions in $C[a,b]$. (Ex!)

Def: Let $d$ and $\rho$ be 2 metrics defined on $X$.

(1) We call $\rho$ is stronger than $d$ or $d$ is weaker than $\rho$,

if $\exists\, C>0$ s.t.

$$\boxed{d(x,y) \leq C\,\rho(x,y), \quad \forall x,y \in X}$$

(2) They are equivalent if $\rho$ is stronger and weaker than $d$.

i.e. $\exists\, C_1, C_2 > 0$ s.t.

$$\boxed{d(x,y) \leq C_1\,\rho(x,y) \leq C_2\,d(x,y) \quad \forall x,y \in X.}$$

(or $c_1 d(x,y) \leq \rho(x,y) \leq c_2 d(x,y)$ )

Prop: (1) If $\rho$ is stronger than $d$, then

$\{x_n\}$ converges in $(X,\rho)$ implies

$\{x_n\}$ converges in $(X,d)$, and have the same limit.

( i.e. $\rho(x_n,y)\to 0 \Rightarrow d(x_n,y)\to 0$ )

(2) If $\rho$ is equivalent to $d$, then $\{x_n\}$ converges in

$(X,\rho)$ if and only if $\{x_n\}$ converges in $(X,d)$.

( i.e. $\rho(x_n,y)\to 0 \Leftrightarrow d(x_n,y)\to 0$ )

(3) "equivalent" of metrics defined above is an equivalent relation.

(Pf: Easy ex! )

eg On $\mathbb{R}^n$, $\begin{cases} d_1(x,y) = \sum_i |x_i - y_i| \\ d_2(x,y) = \left( \sum_i |x_i-y_i|^2 \right)^{1/2} \\ d_\infty(x,y) = \max_i |x_i - y_i| \end{cases}$

Check: (i) $d_2(x,y) \leq \sqrt{n}\, d_\infty(x,y) \leq \sqrt{n}\, d_2(x,y)$

(Ex!)

(ii) $d_1(x,y) \leq n\, d_\infty(x,y) \leq n\, d_1(x,y)$

Therefore, $d_1$, $d_2$ & $d_\infty$ are equivalent metrics on $\mathbb{R}^n$.

(Note: the constants depend on the dimension $n$.)

eg $X = C[a,b]$ $\begin{cases} d_1(f,g) = \int_a^b |f-g| \\ d_\infty(f,g) = \max_{[a,b]} |f-g| \end{cases}$ (infinite dim'l)

Then clearly

$$d_1(f,g) \leq (b-a)\, d_\infty(f,g), \quad \forall f,g \in C[a,b]$$

$\therefore$ $d_\infty$ is stronger than $d_1$.

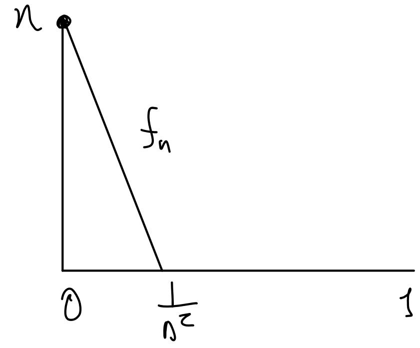
However, it is impossible to find $C>0$ s.t.

$$d_\infty(f,g) \leq C\, d_1(f,g), \quad \forall f,g \in C[a,b]$$

$\therefore$ $d_1$ & $d_\infty$ are not equivalent.

Pf.: Define $f_n$ on $[a,b] = [0,1]$

$$f_n(x) = \begin{cases} -n^3 x + n, & x \in [0, \frac{1}{n^2}] \\ 0, & x \in (\frac{1}{n^2}, 1]. \end{cases}$$




Then $d_1(f_n, 0) = \int_0^1 |f_n| = \frac{1}{2n} \to 0$ as $n \to \infty$

$$d_\infty(f_n, 0) = \max_{[0,1]} |f_n(x)| = n \to \infty \text{ as } n \to \infty$$

Suppose $\exists\, C > 0$ s.t. $d_\infty(f,g) \le C\, d_1(f,g)$, $\forall f, g \in C[0,1]$

Then

$$n = d_\infty(f_n, 0) \le C\, d_1(f_n, 0) = \frac{C}{2n}, \qquad \forall n$$

which is a contradiction by letting $n \to \infty$.

$\therefore$ $d_1$ is <u>not</u> stronger than $d_\infty$.

Therefore $d_1$ & $d_\infty$ are <u>not</u> equivalent. ※

**Def:** Let $f : (X, d) \to (Y, \rho)$ be a mapping between two metric spaces, and $x \in X$. We call

- $f$ is <u>continuous</u> at $x$ if $f(x_n) \to f(x)$ in $(Y, \rho)$ whenever $x_n \to x$ in $(X, d)$.

- It is <u>continuous</u> on a set $E \subset X$ if it is continuous at every point of $E$.

**Prop 2.2** Let $f : (X, d) \to (Y, \rho)$ be a mapping between 2 metric spaces, and $x_0 \in X$. Then

$$\boxed{f \text{ is continuous at } x_0 \iff \begin{cases} \forall \varepsilon > 0, \exists \delta > 0 \text{ such that} \\ \rho(f(x), f(x_0)) < \varepsilon, \ \forall x \text{ with } d(x, x_0) < \delta \end{cases}}$$

(Pf: Ex!)

**Prop 2.3**: Let $f : (X, d) \to (Y, \rho)$ and

$$g : (Y, \rho) \to (Z, m)$$

are mappings between metric spaces.

(a) If $f$ is continuous at $x$ & $g$ is continuous at $f(x)$,

then $g \circ f : (X, d) \to (Z, m)$ is continuous at $x$.

(b) If $f$ is cts in $X$ and $g$ is cts. in $Y$, then

$g \circ f$ is cts in $X$.

(Pf: Easy)