§1.5 Mean Convergence of Fourier Series

Notation:

$R[-\pi,\pi]$ = set of Riemann integrable (real) functions on $[-\pi,\pi]$.

Def: (1) $\forall f, g \in R[-\pi,\pi]$, the L²-product (L² inner product) is given by

$$\boxed{\langle f, g\rangle_2 = \int_{-\pi}^{\pi} f(x)g(x)\,dx}$$

(Note: for cpx function $\langle f, g\rangle_2 = \int_{-\pi}^{\pi} f\bar{g}$)

(2) The L²-norm of $f \in R[-\pi,\pi]$ is $\boxed{\|f\|_2 = \sqrt{\langle f, f\rangle_2}}$

(3) The L²-distance between $f, g \in R[-\pi,\pi]$ is $\boxed{\|f-g\|_2}$.

(4) We said that $f_n \to f$ in L² sense if $\|f_n - f\|_2 \to 0$ as $n\to\infty$.

(i.e. $\lim_{n\to\infty} \int_{-\pi}^{\pi} (f_n - f)^2\,dx = 0$, "mean convergence")

Caution: L²-norm & L²-distance on $R[-\pi,\pi]$ are not really "norm" & "distance" in the strict sense as

$$\begin{cases} \|f\|_2 = 0 \not\Rightarrow f=0 \text{ in } R[-\pi,\pi] \\ \|f-g\|_2 = 0 \not\Rightarrow f=g \text{ in } R[-\pi,\pi] \end{cases}$$

(we only have $\begin{cases} f=0 \text{ almost everywhere} \\ f=g \text{ almost everywhere} \end{cases}$ resp.)

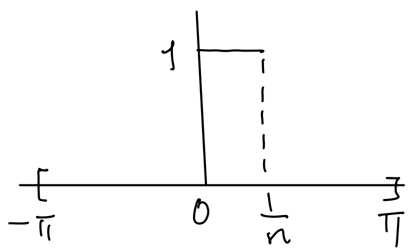
Note : It is not hard to show that

$$f_n \to f \text{ uniformly} \Rightarrow \|f_n - f\|_2 \to 0$$

However $\|f_n - f\|_2 \to 0 \not\Rightarrow f_n \to f$ uniformly !

$$\left(\begin{array}{ll} \text{i.e.} & \|f_n - f\|_\infty \to 0 \Rightarrow \|f_n - f\|_2 \to 0 \\ \text{but} & \|f_n - f\|_2 \to 0 \not\Rightarrow \|f_n - f\|_\infty \to 0 \end{array} \right)$$

eg :



$$f_n(x) = \begin{cases} 1, & x \in [0, \frac{1}{n}] \\ 0, & \text{otherwise} \end{cases}$$

Then $\|f_n\|_2^2 = \int_{-\pi}^{\pi} f_n^2 = \frac{1}{n} \to 0$

$\therefore$ $f_n \to 0$ in L^2-sence

But $f_n \not\to 0$ uniformly .

In fact $f_n(x) \to \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$

(not even pointwise converge to 0, & the pointwise limit is discts.)

Application to Fourier Series

Consider the functions on $[-\pi, \pi]$:

$$\begin{cases} \varphi_0 = \dfrac{1}{\sqrt{2\pi}} & (\text{const. function}) \\ \varphi_n = \dfrac{1}{\sqrt{\pi}} \cos nx & \\ & (n \geq 1) \\ \psi_n = \dfrac{1}{\sqrt{\pi}} \sin nx & \end{cases}$$

Then

$$\begin{cases} \langle \varphi_m, \varphi_n \rangle_2 = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \\ \langle \varphi_m, \psi_n \rangle_2 = 0, \quad \forall m, n \qquad \text{(check!)} \\ \langle \psi_m, \psi_n \rangle_2 = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \end{cases}$$

$\therefore \left\{ \varphi_0 = \frac{1}{\sqrt{2\pi}}, \varphi_n = \frac{1}{\sqrt{\pi}} \cos nx, \psi_n = \frac{1}{\sqrt{\pi}} \sin nx \right\}_{n=1}^{\infty}$

can be regarded as an "orthonormal basis" in $R[-\pi, \pi]$.

<u>Notation</u> : We denote

$$E_N \overset{\text{def}}{=} \text{span}\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx \right\}_{n=1}^{N} \qquad (\dim E_N = 2N+1)$$

= $(2N+1)$ dim'l vector subspace of $R[-\pi, \pi]$ spanned by

the 1^{st} $(2N+1)$ trigonometric functions.

In general, if we have an <u>orthonormal set</u> (or <u>orthonormal family</u>)

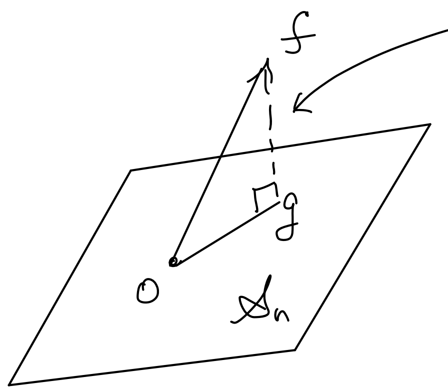
$$\begin{cases} \{\phi_n\}_{n=1}^{\infty} \text{ in } R[-\pi,\pi] \text{ with} \\ \langle \phi_n, \phi_m \rangle_2 = \delta_{mn}, \end{cases}$$

we set $\mathcal{S}_n = \text{span}\langle \phi_1, \cdots, \phi_n \rangle$

$= n$-dim'l subspace spanned by the 1st n functions in the orthonormal set

Then $\forall f \in R[-\pi,\pi]$, one considers the <u>minimization problem</u>

$$\inf\{\|f-g\|_2 : g \in \mathcal{S}_n\}$$



minimize this distance (L^2) over all possible $g \in \mathcal{S}_n$

> <u>Prop 1.14</u>: The unique minimizer of $\inf_{g\in\mathcal{S}_n} \|f-g\|_2$ is attained at the function $g = \sum_{k=1}^{n} \langle f, \phi_k \rangle_2 \, \phi_k \in \mathcal{S}_n$

<u>Pf</u>: Note that minimize $\|f-g\|_2 \iff$ minimize $\|f-g\|_2^2$

Now $\forall g \in S_n$, $g = \sum_{k=1}^{n} \beta_k \phi_k$ for some $\beta_1, \cdots, \beta_n$

and

$$\|f-g\|_2^2 = \int_{-\pi}^{\pi} \left|f - \sum_{k=1}^{n} \beta_k \phi_k\right|^2$$

can be regarded as a function of $\beta = (\beta_1, \cdots, \beta_n)$ and let denote it by

$$\|f-g\|_2^2 = \Phi(\beta_1, \cdots, \beta_n) = \Phi(\beta)$$

We first need to show that

$$\Phi(\beta_1, \cdots, \beta_n) \to \infty \quad \text{as} \quad \|\beta\| = \sqrt{\sum \beta_k^2} \to +\infty .$$

$$\Phi(\beta) = \int_{-\pi}^{\pi} \left(f - \sum_{k=1}^{n} \beta_k \phi_k\right)^2$$

$$= \left(\int_{-\pi}^{\pi} f^2\right) - 2 \sum_{k=1}^{n} \beta_k \left(\int_{-\pi}^{\pi} f \phi_k\right) + \sum_{l,k=1}^{n} \beta_k \beta_l \int_{-\pi}^{\pi} \phi_k \phi_l$$

(orthonormal)

$$= \|f\|_2^2 - 2 \sum_{k=1}^{\infty} \left(\frac{\beta_k}{\sqrt{2}}\right)\left(\sqrt{2}\langle f, \phi_k \rangle_2\right) + \sum_{k=1}^{n} \beta_k^2$$

$$\geq \|f\|_2^2 - \sum_{k=1}^{\infty} \left(\frac{\beta_k^2}{2} + 2\langle f, \phi_k \rangle_2^2\right) + \sum_{k=1}^{n} \beta_k^2$$

$$= \underbrace{\|f\|_2^2 - 2 \sum_{k=1}^{\infty} \langle f, \phi_k \rangle_2^2}_{\text{indep. of } \beta_k} + \frac{1}{2} \sum_{k=1}^{n} \beta_k^2 \to +\infty$$

as $\|\beta\| \to +\infty$.

Clearly, $\Phi(\beta)$ is continuous,

$\therefore$ $\Phi(\beta)$ attains a minimum at some finite point $\beta=(\beta_1,\cdots,\beta_n)$

Then easy calculus

$\Rightarrow$ the unique maximum is given by

$$\beta_k = \langle f, \phi_k \rangle_2 \;, \quad \forall k=1,\cdots,n .$$

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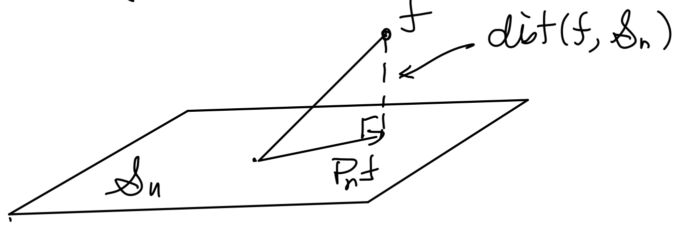
Notes : (1) The minimizer $g=\sum_{k=1}^{n}\langle f,\phi_k\rangle_2\,\phi_k$ of $\|f-g\|_2$ over S_n

is called the orthogonal projection of f onto S_n &

denoted by $P_n f$ $(\in S_n)$.

(2) $\quad \mathrm{dist}(f, S_n)$ $\left(\overset{\text{def}}{=} \inf\{\mathrm{dist}(f,g) : g\in S_n\}\right)$

$= \|f-P_n f\|_2$



Cor 1.15 For 2π-periodic function f integrable on $[-\pi,\pi]$ and $n\geq 1$,

$$\|f-S_N f\|_2 \leq \|f-g\|_2,$$

(nth partial sum of the Fourier series of f)

$\forall g$ of the form

$$g=\alpha_0+\sum_{k=1}^{N}(\alpha_k\cos kx+\beta_k\sin kx)$$

with $\alpha_0, \alpha_k, \beta_k \in \mathbb{R}$.

Pf: By def. of Fourier coefficients $S_nf = P_nf$ of the

$$\text{span}\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos kx, \frac{1}{\sqrt{\pi}}\sin kx\right\}_{k=1}^{n}:$$

$$\begin{cases} a_0 = \langle f, \frac{1}{\sqrt{2\pi}}\rangle_2 \cdot \frac{1}{\sqrt{2\pi}} \\ a_k\cos kx = \langle f, \frac{1}{\sqrt{\pi}}\cos kx\rangle_2 \cdot \frac{1}{\sqrt{\pi}}\cos kx \qquad (\text{Ex!}) \\ b_k\sin kx = \langle f, \frac{1}{\sqrt{\pi}}\sin kx\rangle_2 \cdot \frac{1}{\sqrt{\pi}}\sin kx \end{cases}$$

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Thm 1.16 For 2π-periodic (real) function f (Riemann) integrable on $[-\pi,\pi]$,

$$\boxed{\lim_{n\to\infty}\|S_nf-f\|_2=0}$$

i.e. the nth partial sum of the Fourier Series of f converges to f in L^2-sense.