

1. Each of the following functions are defined on $(-\pi, \pi]$.
Sketch the 2π -periodic extension, find the corresponding Fourier expansion, and discuss the pointwise convergence (using Thm 1.5 and 1.6)

$$(a) \quad f_1(x) = \begin{cases} 0, & x \in [0, \pi] \\ x, & x \in (-\pi, 0) \end{cases}$$

$$(b) \quad f_2(x) = \begin{cases} 2, & x \in [0, \pi] \\ -1, & x \in (-\pi, 0) \end{cases}$$

$$(c) \quad f_3(x) = e^{3x}$$

(2) Let $f: [a, b] \rightarrow \mathbb{R}$ & $g: [a, b] \rightarrow \mathbb{R}$ be bounded functions satisfying Lipschitz condition.

(a) Show that $fg: [a, b] \rightarrow \mathbb{R}$, the product function, satisfies Lipschitz condition.

(b) If $f[a, b] \subset [a, b]$, then $g \circ f: [a, b] \rightarrow \mathbb{R}$ also satisfies Lipschitz condition.

(3) Consider the function $f(x) = -x$ on $(0, 2\pi]$ and its 2π -periodic extension $\tilde{f}(x)$ by $\tilde{f}(x) = f(x - 2k\pi)$ for $x \in (2k\pi, 2(k+1)\pi]$, $\forall k \in \mathbb{Z}$.

Sketch \tilde{f} , find its Fourier series, and discuss the pointwise convergence. Finally, if the Fourier series converges at the point $x=0$, what value does it limit to?

(4) Consider the same function $f(x) = -x$ but only on $[0, \pi]$. Extending it to an even function $f_1(x)$ on $[-\pi, \pi]$, then further extend f_1 to a 2π -periodic function \tilde{f}_1 as usual. Sketch \tilde{f}_1 , find its Fourier series, and discuss the pointwise convergence. Finally, if the Fourier series converges at the point $x=0$, what value does it limit to? Is it the same value as in the problem (3)?

(End)