Proof of Thm 1.5

Let f be Lip cts. at a point $x_0 \in [-\pi, \pi]$.

Step 1 $(S_n f)(x_0) = a_0 + \sum_{k=1}^{n} (a_k \cos kx_0 + b_k \sin kx_0)$

$$= \int_{-\pi}^{\pi} D_n(z) f(x_0+z)\, dz$$

where

$$D_n(z) = \begin{cases} \dfrac{\sin (n+\frac{1}{2}) z}{2\pi \sin \frac{1}{2} z}, & \text{if } z \neq 0 \\[2ex] \dfrac{2n+1}{2\pi}, & \text{if } z = 0, \end{cases}$$

is called the Dirichlet kernel.

Pf:

$$(S_n f)(x_0) = a_0 + \sum_{k=1}^{n} (a_k \cos kx_0 + b_k \sin kx_0)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,dy + \sum_{k=1}^{n} \left[\left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos ky\, dy \right) \cos kx_0 + \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin ky\, dy \right) \sin kx_0 \right]$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{k=1}^{n} (\cos ky \cos kx_0 + \sin ky \sin kx_0) \right] f(y)\,dy$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{k=1}^{n} \cos k(y - x_0) \right] f(y)\,dy$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{k=1}^{n} \cos kz \right] f(z+x_0)\,dz \qquad \left(\begin{array}{c} z = y - x_0 \text{ \&} \\ 2\pi\text{-periodic} \end{array} \right)$$

Since $\frac{1}{2} + \sum_{k=1}^{n} \cos kz = \frac{\sin(n+\frac{1}{2})z}{2\sin(\frac{1}{2}z)}$ for $z \neq 0$ (Check "$z=0$" case (Ex!))

(Ex: Calculate $e^{-in\theta} + \cdots + 1 + \cdots + e^{in\theta}$

and use $1+z+\cdots+z^k = \frac{z^{k+1}-1}{z-1}$)

$$(S_n f)(x_0) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})z}{2\sin\frac{1}{2}z} f(x_0+z)\,dz$$

$$= \int_{-\pi}^{\pi} D_n(z) f(x_0+z)\,dz \qquad \text{※}$$

Step 2 (Properties of $D_n(z)$)

(1) $\int_{-\pi}^{\pi} D_n(z)\,dz = 1$

(2) $D_n(z)$ is even, cts, 2π-periodic on $[-\pi,\pi]$

and $D_n\left(\frac{2k\pi}{2n+1}\right) = 0$ for $k=-n,\cdots,0,\cdots,n$

(3) $\max_{[-\pi,\pi]} D_n(z) = D_n(0) = \frac{2n+1}{2\pi}$

(4) $\forall\ 0<\delta<\pi/2$, $\int_0^{\delta} |D_n(z)|\,dz \to +\infty$ as $n \to +\infty$.

Pf (1) Easy: by integrating $\int_{-\pi}^{\pi} \left(\frac{1}{2} + \sum_{k=1}^{n} \cos kz\right) dz$

(2) & (3) are easy exercises.

(4) Let $0<\delta<\pi/2$

Then $\forall n \in \mathbb{N}$, $\exists N \in \mathbb{N}$ s.t.

$$N < \frac{(n+\frac{1}{2})\delta}{\pi} \leq N+1$$

Clearly $N \to +\infty$ as $n \to +\infty$.

Now $$\int_0^\delta |D_n(z)|\,dz = \int_0^\delta \frac{|\sin(n+\frac{1}{2})z|}{2\pi\,|\sin\frac{z}{2}|}\,dz$$

$$= \int_0^{(n+\frac{1}{2})\delta} \frac{|\sin t|}{2\pi\,|\sin\frac{t}{2n+1}|}\,\frac{2\,dt}{(2n+1)} \qquad \left(t=(n+\tfrac{1}{2})z\right)$$

$$= \frac{1}{\pi}\int_0^{(n+\frac{1}{2})\delta} \frac{|\sin t|}{t}\cdot\frac{\left(\frac{t}{2n+1}\right)}{|\sin\frac{t}{2n+1}|}\,dt$$

$$\geq \frac{1}{\pi}\int_0^{(n+\frac{1}{2})\delta} \frac{|\sin t|}{t}\,dt \qquad \left(\text{Using } \frac{|\sin x|}{x}<1 \text{ for } 0<x\right)$$

$$\geq \frac{1}{\pi}\int_0^{\pi N} \frac{|\sin t|}{t}\,dt$$

$$= \frac{1}{\pi}\sum_{k=1}^{N}\int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{t}\,dt$$

$$= \frac{1}{\pi}\sum_{k=1}^{N}\int_0^{\pi} \frac{|\sin s|}{s+(k-1)\pi}\,ds \qquad (s=t-(k-1)\pi)$$

$$\geq \frac{1}{\pi} \sum_{k=1}^{N} \int_0^{\pi} \frac{|\sin s|}{k\pi} ds \qquad (s+(k-1)\pi = t \leq k\pi)$$

$$= \frac{1}{\pi^2} \left(\int_0^{\pi} |\sin s| ds \right) \sum_{k=1}^{N} \frac{1}{k}$$

$$= \frac{2}{\pi^2} \sum_{k=1}^{N} \frac{1}{k}$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, and $N \to \infty$ as $n \to \infty$,

we have $\qquad \lim_{n\to\infty} \int_0^{\delta} |D_n(z)| dz = +\infty$

$\divideontimes$

<u>Step 3</u> Splitting $(S_n f)(x_0) - f(x_0) = \mathrm{I} + \mathrm{II}$ into integrals concentrated in $[-\delta, \delta]$ & (essentially) outside $[-\delta, \delta]$.

By (1) in step 2, $\qquad f(x_0) = \int_{-\pi}^{\pi} D_n(z) f(x_0) dz$

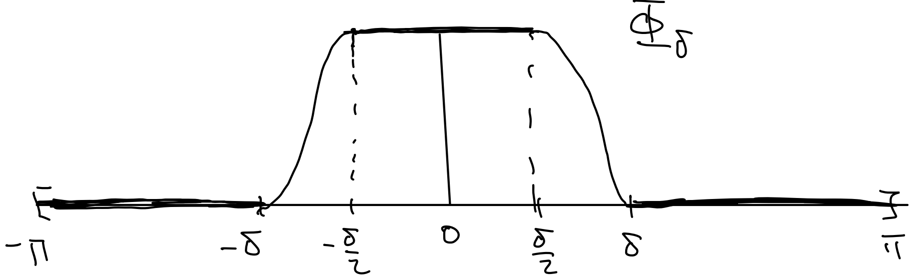
$$\therefore \quad (S_n f)(x_0) - f(x_0) = \int_{-\pi}^{\pi} D_n(z) [f(x_0+z) - f(x_0)] dz$$

Let Φ_δ be a "cut-off" function s.t.

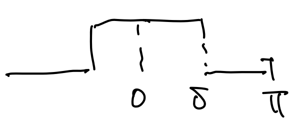
(i) Φ_δ is cts & $0 \leq \Phi_\delta \leq 1$

(ii) $\Phi_\delta(t) \equiv 1$ for $|t| \leq \frac{\delta}{2}$

(iii) $\Phi_\delta(t) \equiv 0$ for $|t| \geq \delta$.



(This is used because of the proof of Thm 1.7 is in mind (which will be omitted)



is enough if we only want to prove Thm 1.5)

Then $(S_n f)(x_0) - f(x_0)$

$$= \int_{-\pi}^{\pi} D_n(z)\,[f(x_0+z) - f(x_0)]\,dz$$

$$= \int_{-\pi}^{\pi} \Phi_\delta(z) D_n(z)\,[f(x_0+z) - f(x_0)]\,dz$$
$$+ \int_{-\pi}^{\pi} (1-\Phi_\delta(z))\, D_n(z)\,[f(x_0+z) - f(x_0)]\,dz$$

$$= \mathrm{I} + \mathrm{II},$$

where $\mathrm{I} = \int_{-\pi}^{\pi} \Phi_\delta(z) D_n(z)\,[f(x_0+z) - f(x_0)]\,dz$

$$= \int_{-\delta}^{\delta} \Phi_\delta(z) D_n(z)\,[f(x_0+z) - f(x_0)]\,dz$$

$$\mathrm{II} = \int_{-\pi}^{\pi} (1-\Phi_\delta(z))\, D_n(z)\,[f(x_0+z) - f(x_0)]\,dz$$

$$= \left(\int_{-\pi}^{-\delta/2} + \int_{\delta/2}^{\pi}\right) (1-\Phi_\delta(z))\, D_n(z)\,[f(x_0+z) - f(x_0)]\,dz$$

Step 4: $\exists\, L > 0$ and $\delta_2 > 0$ such that

$$|I| \leq \frac{4\delta L}{\pi}, \quad \forall\, 0 < \delta < \delta_2.$$

Pf: By Lip cts. at x_0, $\exists\, L > 0$ & $\delta_0 > 0$ s.t.

$$|f(x_0+z) - f(x_0)| \leq L|z|, \quad \forall\, |z| < \delta_0$$

Since $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$, $\exists\, \delta_1 > 0$ s.t.

$$\left| \frac{\sin\frac{z}{2}}{\frac{z}{2}} \right| > \frac{1}{2}, \quad \forall\, |z| < \delta_1$$

Therefore, for $\delta_2 = \min\{\delta_0, \delta_1\} > 0$

$$\frac{|f(x_0+z) - f(x_0)|}{|\sin\frac{z}{2}|} \leq \frac{L|z|}{\frac{1}{2}|\frac{z}{2}|} = 4L, \quad \forall\, |z| < \delta_2$$

Hence $\forall\, 0 < \delta < \delta_2$, we have

$$|I| \leq \int_{-\delta}^{\delta} |\Phi_\delta(z)|\, |D_n(z)|\, |f(x_0+z) - f(x_0)|\, dz$$

$$= \int_{-\delta}^{\delta} |\Phi_\delta(z)| \frac{|\sin(n+\frac{1}{2})z|}{2\pi\,|\sin\frac{z}{2}|} |f(x_0+z) - f(x_0)|\, dz$$

$$\leq \int_{-\delta}^{\delta} 1 \cdot \frac{1}{2\pi} \cdot 4L\, dz = \frac{4\delta L}{\pi} \quad \#$$

Step5 $\forall \varepsilon>0$, $\exists \delta>0$ & $n_0>0$ s.t.

$$\frac{4\delta L}{\pi} < \frac{\varepsilon}{2} \quad \text{and} \quad |\mathrm{II}| < \frac{\varepsilon}{2}, \quad \forall n \geq n_0.$$

Pf: $\forall \varepsilon>0$, we take $\delta = \min\left\{\frac{\varepsilon\pi}{8L}, \delta_2\right\} > 0$ (L & δ_2 as in step 4)

Then $\frac{4\delta L}{\pi} < \frac{\varepsilon}{2}$,

and for this fixed $\delta>0$,

$$\mathrm{II} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1-\Phi_\delta(z)\right)\cdot\frac{\sin(n+\frac{1}{2})z}{\sin\frac{z}{2}}\left[f(x_0+z)-f(x_0)\right]dz$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-\Phi_\delta(z))\left[f(x_0+z)-f(x_0)\right]}{\sin\frac{z}{2}}\left(\sin nz\cos\frac{z}{2}+\cos nz\sin\frac{z}{2}\right)dz$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{(1-\Phi_\delta(z))\left[f(x_0+z)-f(x_0)\right]}{2\sin\frac{z}{2}}\cos\frac{z}{2}\right]\sin nz\,dz$$

$$+\frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{(1-\Phi_\delta(z))\left[f(x_0+z)-f(x_0)\right]}{2\cancel{\sin\frac{z}{2}}}\cancel{\sin\frac{z}{2}}\right]\cos nz\,dz$$

$$= b_n(F_1)+a_n(F_2) \qquad \text{(Fourier coefficients of } F_1 \text{ \& } F_2\text{)}$$

where $F_1(z) = \frac{(1-\Phi_\delta(z))\left[f(x_0+z)-f(x_0)\right]}{2\sin\frac{z}{2}}\cos\frac{z}{2}$

$$F_2(z) = \frac{(1-\Phi_\delta(z))\left[f(x_0+z)-f(x_0)\right]}{2}$$

$F_2(z)$ is clearly integrable on $[-\pi, \pi]$.

For $F_1(z)$, note that $1-\Phi_\delta(z) = 0$ on $[-\frac{\delta}{2}, \frac{\delta}{2}]$

& $|\sin\frac{z}{2}| \geq \sin\frac{\delta}{4} > 0$ for $\frac{\delta}{2} \leq |z| \leq \pi$

$\Rightarrow F_1(z)$ is also integrable on $[-\pi, \pi]$.

Therefore Riemann-Lebesgue Lemma implies

$$\left.\begin{array}{l} b_n(F_1) \\ a_n(F_2) \end{array}\right\} \to 0 \quad \text{as } n \to \infty$$

$\therefore \exists n_0 > 0$ s.t. $|b_n(F_1)|$ & $|a_n(F_2)| < \frac{\varepsilon}{4}$, $\forall n \geq n_0$

$$\Rightarrow \quad |II| \leq |b_n(F_1)| + |a_n(F_2)| < \frac{\varepsilon}{2}$$

※

<u>Final Step</u> By Steps 3, 4 & 5, we have

$\forall \varepsilon > 0$, $\exists n_0 > 0$ s.t.

$$|(S_n f)(x_0) - f(x_0)| = |I + II|$$

$$\leq |I| + |II| \leq \frac{4\delta L}{\pi} + \frac{\varepsilon}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n \geq n_0$$

$\therefore (S_n f)(x_0) \to f(x_0)$ as $n \to \infty$.

※

§1.4 Weierstrass Approximation Theorem (Application of Thm 1.7)

Recall : A cts function g defined on $[a,b]$ is piecewise linear

if $\exists$ a partition $a = a_0 < a_1 < \cdots < a_n = b$ such that

g is linear on each subinterval $[a_j, a_{j+1}]$.

Prop 1.11 Let f be a cts function on $[a,b]$. Then $\forall \varepsilon > 0$,
$\exists$ a cts, piecewise linear g with

$$\begin{cases} g(a) = f(a), \quad g(b) = f(b) \text{ such that} \\ \|f - g\|_\infty < \varepsilon \end{cases}$$

$$\left(\|f-g\|_\infty = \sup_{[a,b]} |f(x) - g(x)| . \right)$$

Pf: f cts on closed interval $[a,b]$

$\Rightarrow$ f uniform cts on $[a,b]$

$\Rightarrow$ $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

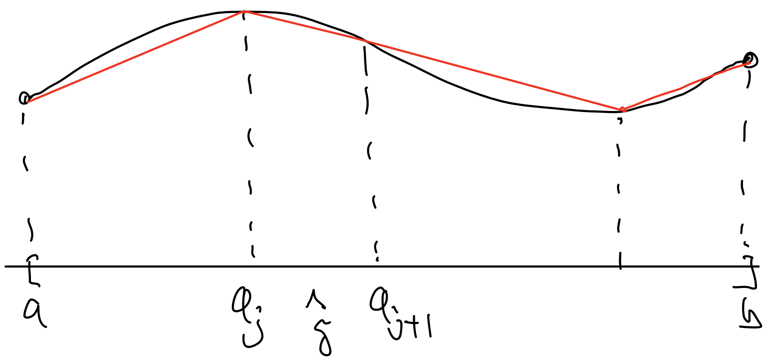
$$|f(x) - f(y)| < \varepsilon/2, \quad \forall |x-y| < \delta \qquad (x, y \in [a,b])$$

Partition $[a,b]$ into subintervals $I_j = [a_j, a_{j+1}]$

s.t. $\quad |I_j| = a_{j+1} - a_j < \delta, \ \forall j.$

Define

$$g(x) = \frac{f(a_{j+1}) - f(a_j)}{a_{j+1} - a_j}(x - a_j) + f(a_j), \quad \forall x \in I_j.$$



Clearly $g(a_j) = f(a_j)$, $\forall j$.

In particular $g(a) = f(a)$ & $g(b) = f(b)$.

And $g(x)$ is piecewise linear on $[a, b]$ (and cts)

Also $\forall x \in I_j \subset [a, b]$

$$|f(x) - g(x)| = \left| f(x) - \frac{f(a_{j+1}) - f(a_j)}{a_{j+1} - a_j}(x - a_j) - f(a_j) \right|$$

$$\leq |f(x) - f(a_j)| + |f(a_{j+1}) - f(a_j)| \cdot \frac{x - a_j}{a_{j+1} - a_j}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore \sup_{\cup I_j} |f(x) - g(x)| < \varepsilon$, i.e. $\|f - g\|_\infty < \varepsilon$.

#

Terminology: A trigonometric polynomial is of the form

$$P(\cos x, \sin x)$$

where $P(x,y)$ is a polynomial of 2-variables

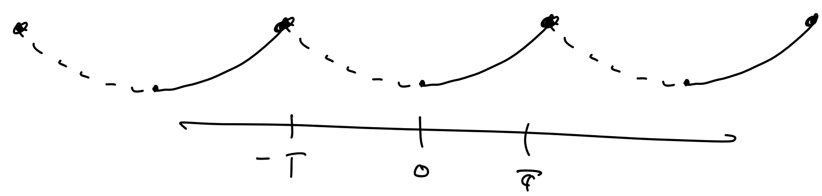
Note: A trigonometric polynomial is a finite Fourier series and vice-versa (Ex!)

Prop 1.12 Let f be a cts function on $[0,\pi]$.

Then $\forall \varepsilon > 0$ $\exists$ a trigonometric polynomial h

s.t. $\|f - h\|_\infty < \varepsilon$.

Pf: Extend f to $[-\pi,\pi]$ by



$$f(x) = \begin{cases} f(x), & x \in [0,\pi] \\ f(-x), & x \in [-\pi, 0] \end{cases}$$

(even extension)

Then this extension is cts on $[-\pi,\pi]$ & $f(\pi) = f(-\pi)$,

hence extends to a 2π-periodic cts. function on $\mathbb{R}$

By Prop 1.11, $\forall \varepsilon > 0$, $\exists$ piecewise linear (cts) g on $[-\pi,\pi]$

s.t. $\|f-g\|_\infty < \varepsilon/2$ (sup taking over $[-\pi,\pi]$)

and $g(\pi) = f(\pi) = f(-\pi) = g(-\pi)$.

$\Rightarrow$ g extends to a piecewise linear 2π-periodic function $\tilde{g}$ on $\mathbb{R}$.

Clearly $\tilde{g}$ satisfies a Lip condition (Check!)

Then Thm 1.7 $\Rightarrow$ $\exists\, N > 0$ s.t

$$\|g - S_N g\|_\infty < \varepsilon/2 \qquad (S_N g \to g \text{ uniformly})$$

Therefore,

$$\|f - S_N g\|_\infty \le \|f-g\|_\infty + \|g - S_N g\|_\infty$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore$ $h = S_N g$ is the required trigonometric polynomial. #

Thm 1.13 (Weierstrass Approximation Theorem)

Let $f \in C[a,b]$. Then $\forall \varepsilon > 0$, $\exists$ a polynomial q s.t.

$$\|f - q\|_\infty < \varepsilon.$$

Pf: Consider $[a,b] = [0,\pi]$ first.

Extend f to $[-\pi, \pi]$ as in Prop 1.12.

$\forall \varepsilon > 0$, choose trigonometric polynomial $h = P(\cos x, \sin x)$

s.t.

$$\| f - h \|_\infty < \varepsilon/2$$

Using the fact that

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$$

converge uniformly.

$\exists N > 0$ s.t.

$$\left\| h(x) - P\left(\sum_{n=0}^{N} \frac{(-1)^n x^{2n}}{(2n)!}, \sum_{n=1}^{N} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \right) \right\|_\infty < \frac{\varepsilon}{2}$$

Clearly $g(x) = P\left(\sum_{n=0}^{N} \frac{(-1)^n x^{2n}}{(2n)!}, \sum_{n=1}^{N} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \right)$

is the required polynomial s.t. $\| f - g \|_\infty < \varepsilon$.

For general $[a,b]$, $\varphi(x) = f\left(\frac{b-a}{\pi} x + a\right) \in C[0, \pi]$

$\Rightarrow \exists\, g(x)$ polynomial s.t. $\| \varphi(x) - g(x) \|_\infty < \varepsilon$ on $[0, \pi]$.

$\Rightarrow g\left(\frac{\pi}{b-a}(x-a)\right)$ is the polynomial s.t.

$$\left\| f(x) - g\left(\frac{\pi}{b-a}(x-a)\right) \right\|_\infty < \varepsilon \quad . \#$$