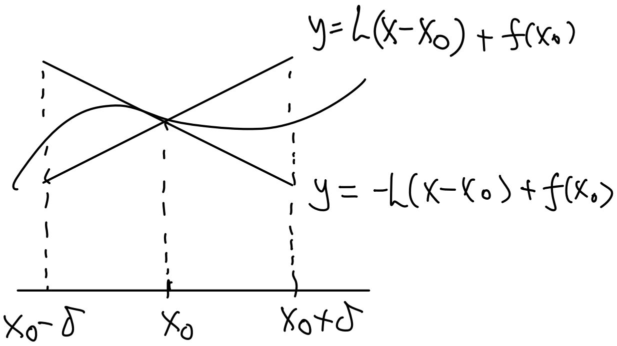
§1.3 Convergence of Fourier Series

Terminology: For $f \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

we denote $(S_n f)(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$

the <u>n-th partial sum</u> of the Fourier Series of f.

> Def: Let f be a function on $[a,b]$. Then f is called <u>Lipschitz continuous at x_0</u> $\in [a,b]$ if $\exists\, L > 0$ & $\delta > 0$ such that
>
> $$|f(x) - f(x_0)| \le L|x - x_0|, \quad \forall\, |x - x_0| < \delta \quad (x \in [a,b])$$



Notes (1) Both L & δ may depend on x_0

(2) If f is Lipschitz continuous at $x_0 \in [a,b]$ & f is <u>bounded</u> on $[a,b]$,

then $\exists\, L' > 0$ (L' may depends on x_0) s.t.

$$|f(x) - f(x_0)| \le L'|x - x_0|, \quad \forall\, x \in [a,b].$$

Pf: By defn, f Lip. cts. at x_0

$\Rightarrow \exists\, L > 0, \delta > 0$ s.t

$$|f(x) - f(x_0)| \le L|x - x_0|, \quad \forall\, |x - x_0| < \delta$$

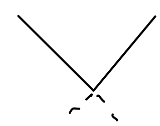
If $|x-x_0| \geq \delta$, then $\dfrac{|x-x_0|}{\delta} \geq 1$

$$\Rightarrow \quad |f(x)-f(x_0)| \leq |f(x)| + |f(x_0)| \leq 2M \quad \text{where } M = \sup_{[a,b]} |f|,$$
$$\leq \frac{2M}{\delta} |x-x_0|$$

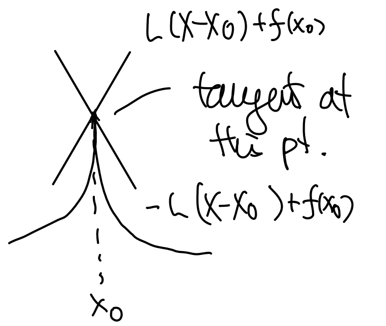
Hence
$$|f(x)-f(x_0)| \leq \begin{cases} L|x-x_0|, & |x-x_0|<\delta \\ \frac{2M}{\delta}|x-x_0|, & |x-x_0| \geq \delta \end{cases}$$

$$\Rightarrow \quad |f(x)-f(x_0)| \leq L'(x-x_0), \quad \forall x \in [a,b],$$
where $L' = \max\{L, \frac{2M}{\delta}\} > 0$ ※

eg: $f \in C^1[a,b]$ (continuously differentiable on $[a,b]$)
$\Rightarrow$ f is Lip. cts. at every $x_0 \in [a,b]$.
On the other hand $f(x) = |x|$ is Lip. cts. at $x=0$, but not differentiable (Ex!)



eg:



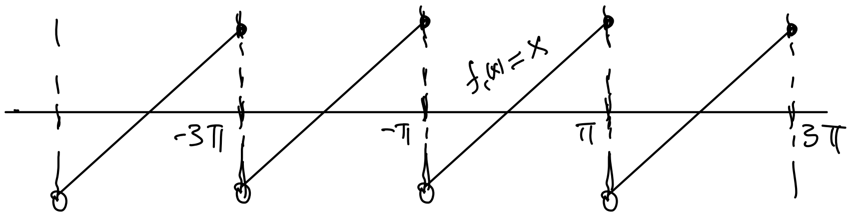
This graph gives a cts function at x_0, but not Lip. cts at x_0.
Explicity example: $f(x) = |x|^\alpha$ with $0<\alpha<1$ is not Lip. cts. at $x=0$.

Thm 15 Let f be a 2π-periodic function integrable on $[-\pi,\pi]$. Suppose that f is Lipschitz continuous at x. Then $\{S_n f(x)\}$ converges to $f(x)$ as $n \to \infty$.

(Pf: later at the end of this section)

eg of application

Recall $f_1(x) = x$ on $[-\pi, \pi]$



Fourier series $\quad x \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$.

It is clear that $f_1(x)$ is Lip. cts at any $x \in (-\pi, \pi)$

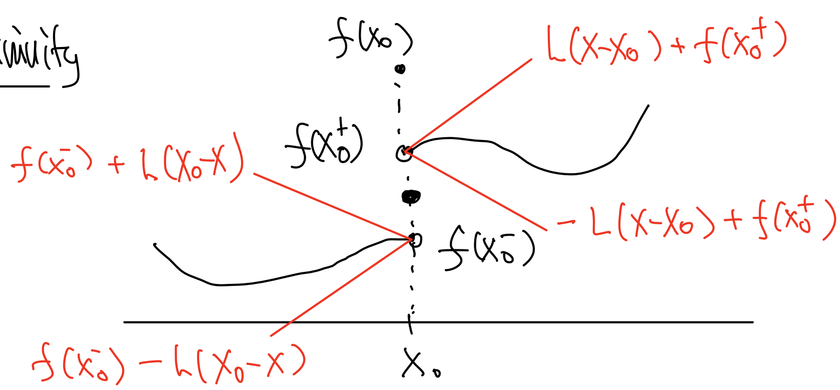
$$\therefore \quad \lim_{N\to+\infty} 2\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} \sin nx = x, \quad \forall x \in (-\pi, \pi)$$

On the other hand, $\tilde{f}_1$ is discontinuous at $x = \pm\pi$

and we've seen that (eg 1.1)

$$\tilde{f}_1(\pm\pi) \neq 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

Jump discontinuity



Thm 1.6 Let f be a 2π-periodic function integrable on $[-\pi,\pi]$.

Suppose that for $x_0 \in [-\pi,\pi]$,

(i) $f(x_0^+) = \lim\limits_{x\to x_0^+} f(x)$ right-hand limit

$f(x_0^-) = \lim\limits_{x\to x_0^-} f(x)$ left-hand limit

} both exist.

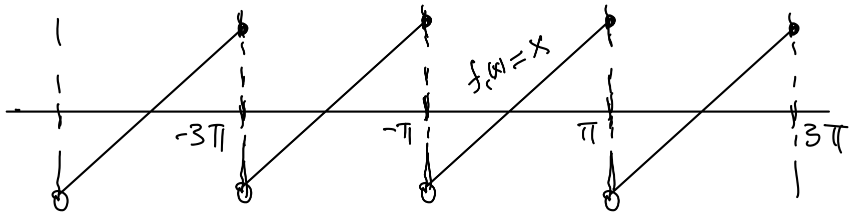
(ii) $\exists\, L>0$ and $\delta>0$ such that

$$\begin{cases} |f(x)-f(x_0^+)| \le L(x-x_0), & 0<x-x_0<\delta \\ \& \ |f(x)-f(x_0^-)| \le L(x_0-x), & 0<x_0-x<\delta \end{cases}$$

Then $S_n f(x_0) \longrightarrow \dfrac{f(x_0^+)+f(x_0^-)}{2}$ as $n\to+\infty$

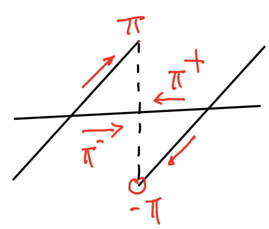
(Pf: Omitted)

<u>eg of application</u> $f_1(x) = x$ on $[-\pi,\pi]$



At $x_0=\pi$, $\tilde{f}_1(x)$ is discontinuous.

(i) $\tilde{f}(\pi^+) = \lim\limits_{x\to\pi^+} \tilde{f}_1(x) = -\pi$



$\tilde{f}(\pi^-) = \lim\limits_{x\to\pi^-} \tilde{f}_1(x) = \pi$

(ii) For $0<x-x_0<\frac{\pi}{2}$ (i.e. $0<x-\pi<\frac{\pi}{2}=\delta$)

we have $|\tilde{f}(x)-\tilde{f}(\pi^+)| = |\tilde{f}(x-2\pi)-(-\pi)|$ L=1

$= |x-2\pi+\pi| = x-\pi \le L(x-\pi)$

Similarly for $0<x_0-x<\frac{\pi}{2}$

Hence conditions of Thm 1.6 are satisfied

$\Rightarrow$ Fourier series $\underset{\displaystyle 0}{\underset{\parallel}{S_n f(\pi)}} \longrightarrow \dfrac{f(\pi^+)+f(\pi^-)}{2} = \dfrac{-\pi+\pi}{2} = 0$.

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Next we turn to "uniform" convergence and need

Def: A function f defined on $[a,b]$ is called to satisfy a Lipschitz condition if $\exists L>0$ such that
$$|f(x)-f(y)| \leq L|x-y|, \quad \forall x,y \in [a,b].$$

Notes: (1) $L>0$ is independent of $x,y\in[a,b]$, a kind of "uniform" Lip condition.

(2) f satisfies a Lip. condition $\Rightarrow$ f is Lip. cts at every point on $[a,b]$.

eg.: If $f\in C^1[a,b] \Rightarrow |f(x)-f(y)| = \left|\int_x^y f'(t)dt\right|$
$$\leq M|y-x|, \quad \forall x,y\in[a,b].$$

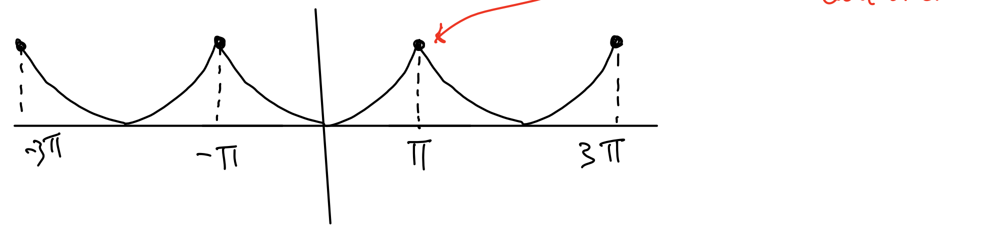
where $M = \sup_{[a,b]} |f'|$.

On the other hand, $f(x)=|x|$ satisfies a Lip condition, but not C^1.

Thm 1.7 Let f be a 2π-periodic function satisfying a Lipschitz condition. Then its Fourier Series converges uniformly to f itself.

(Pf: Omitted)

eg of application $f_2(x) = x^2$ on $[-\pi, \pi]$



$\tilde{f}_2$ satisfies a Lip. condition (Check! (Ex))

$$\Rightarrow \quad \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx \quad \text{converges uniformly to}$$

$f_2(x) = x^2$ on $[-\pi, \pi]$.

(Ex: Put $x=0$ and get $\displaystyle \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$)