Notes: (1) For $x = \pm\pi$, Fourier series of $f_1(x) = x \Big|_{\pm\pi} = 0$

(for eg 1.1)

But $\left.\begin{array}{l} f_1(\pm\pi) = \pm\pi \\ \tilde{f}_1(\pm\pi) = \pi \end{array}\right\} \neq$ Fourier Series of $f_1(x) = x \Big|_{\pm\pi} = 0$

(2) Convergence is not clear for $x \neq \pm\pi$ as the terms decay like $\frac{1}{n}$ & $\sum \frac{1}{n}$ doesn't converge.

Notation "Big O" & "small o"

Let $\{x_n\}$ be a sequence, then

(i) $x_n = O(n^s) \iff |x_n| \leq C n^s$ for some const. $C > 0$

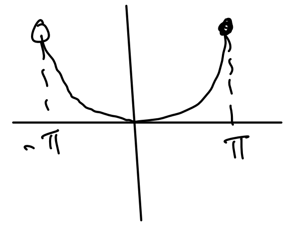
(ii) $x_n = o(n^s) \iff |x_n| / n^s \to 0$ as $n \to \infty$.

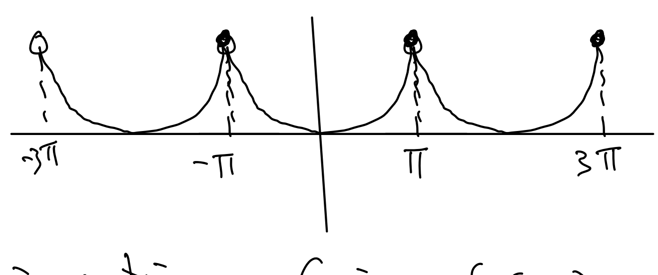
egs: (i) $x_n = \frac{2(-1)^{n+1}}{n} \sin nx = O(n^{-1}) = O(\frac{1}{n})$, $\left(|x_n| \leq \frac{2}{n}\right)$

(ii) $x_n = \log n = o(n)$ $\left(\frac{\log n}{n} \to 0 \text{ as } n \to \infty\right)$

Eg 1.2 $f_2(x) = x^2$ restricted to $(-\pi, \pi]$

Extension to a 2π-periodic function $\tilde{f}_2$ on $\mathbb{R}$





$\tilde{f}_2$ is continuous (since $f_2(-\pi) = f_2(\pi)$)

$\tilde{f}_2$ is an even function

It is an <u>easy exercise</u> of integration to find that

$$f_2(x) = x^2 \sim \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx \qquad (\text{Ex!})$$

(cosine series, f_2 even)

One sees that $\quad a_n = O\left(\frac{1}{n^2}\right) \Rightarrow \sum |a_n| < \infty$

$\Rightarrow$ Fourier series converges uniformly to a continuous function.

(Will it be the function $\tilde{f}_2$? See later discussion)

<u>Observation</u>: Egs 1 & 2 : $\begin{cases} \text{odd function} \rightarrow \text{sine series} \\ \text{even function} \rightarrow \text{cosine series} \end{cases}$

This is true in general! (Ex!)

Complex Fourier Series

Def: (1) A complex trigonometric series is a series of the form

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

($\{c_n\}_{-\infty}^{\infty}$ is called a bisequence of cpx numbers &

$\{c_n e^{inx}\}_{n=-\infty}^{\infty}$ is a bisequence of cpx-valued functions)

(2) $\sum_{-\infty}^{\infty} c_n e^{inx}$ is said to be convergent at x if

$$\lim_{N \to +\infty} \sum_{n=-N}^{N} c_n e^{inx} \text{ exists.}$$

Def: Complex Fourier Series of a 2π-periodic cpx-valued function f which is integrable on $[-\pi, \pi]$, denoted by

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

is a cpx trigonometric series with (cpx) Fourier coefficients c_n defined by

$$\boxed{c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx, \quad \forall n \in \mathbb{Z}}$$

Motivation for cpx Fourier Series:

"If" $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$ & "converges nicely"

Then $f(x) e^{-imx} = \sum_{-\infty}^{\infty} c_n e^{i(n-m)x}$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) e^{-imx}\, dx = \sum_{-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-m)x}\, dx$$

It is easy to find
$$\int_{-\pi}^{\pi} e^{i(n-m)x}\,dx = \begin{cases} 2\pi, & \text{if } n=m \\ 0, & \text{if } n\neq m \end{cases}$$

$$\Rightarrow \quad \int_{-\pi}^{\pi} f(x)e^{-imx}\,dx = c_m \cdot 2\pi \quad \#$$

<u>Relationship</u> between (Real) Fourier Series & Cpx Fourier Series for a <u>real-valued</u> function <u>f</u>.

By $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i\sin nx)\,dx$$

$$= \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\right) - \frac{1}{2}i\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right)$$

Therefore

for $n=0$, $\quad c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = a_0$

for $n\geq 1$, $\quad c_n = \frac{a_n}{2} - i\frac{b_n}{2}$

for $n\leq -1$, then $(-n)\geq 1$ &

$$c_n = \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos((-n)x)\,dx\right) + \frac{1}{2}i\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(-n)x\,dx\right)$$

$$= \frac{1}{2}a_{(-n)} + i\,\frac{1}{2}b_{(-n)}$$

$\therefore$

$$\boxed{c_n = \begin{cases} \frac{1}{2}(a_n - ib_n), & \text{for } n\geq 1 \\ a_0, & \text{for } n=0 \\ \frac{1}{2}(a_{(-n)} + ib_{(-n)}), & \text{for } n\leq -1 \end{cases}}$$

for real-valued function.

Corollary: If f is a real-valued function, then

$$C_{-n} = \overline{C_n} \quad \text{cpx conjugate}, \ \forall n \in \mathbb{Z}$$

$$(\text{i.e. } C_n = \overline{C_{-n}})$$

(Pf: Easy)

Prop: Let f be a 2π-periodic real-valued function which is differentiable on $[-\pi,\pi]$ with f' integrable on $[-\pi,\pi]$.

Denote the Fourier coefficients of f & f' by

$\{a_n(f), b_n(f) ; C_n(f)\}$ & $\{a_n(f'), b_n(f') ; C_n(f')\}$ respectively.

Then

$$\boxed{\begin{aligned} a_n(f') &= n\, b_n(f) \\ b_n(f') &= -n\, a_n(f) \\ \& \quad C_n(f') &= in\, C_n(f) \end{aligned}}$$

(So it is more convenient to work with cpx Fourier coefficients when derivatives involved.)

Pf:

$$a_n(f') = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx \, dx$$

(integration by part)
$$= \frac{1}{\pi}\left[f(x)\cos nx \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)(-n\sin nx)\,dx \right]$$

$(f(\pi) = f(-\pi))$
$$= \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx = n\, b_n(f)$$

Similarly for $b_n(f') = -n\, a_n(f)$ (Check!)

For $C_n(f')$, either from the above formula relating C_n to a_n & b_n, or integration by part directly

$$C_n(f') = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}\,dx = \frac{1}{2\pi}\left[\cancelto{0}{f(x)e^{-inx}\Big|_{-\pi}^{\pi}} + (in)\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \right] = in\, C_n(f)$$

Remarks: (1) f is differentiable on $[-\pi,\pi]$ doesn't implies f' is (Riemann) integrable on $[-\pi,\pi]$. So the conditions in the Prop are needed.

A counterexample can be constructed from the example

$$g(x) = \begin{cases} x^{\frac{3}{2}} \sin\frac{1}{x}, & x>0 \\ 0, & x=0 \end{cases}$$

Then
$$g'(x) = \begin{cases} \frac{3}{2}x^{\frac{1}{2}} \sin\frac{1}{x} - \frac{1}{x^{1/2}}\cos\frac{1}{x}, & x>0 \\ 0, & x=0. \end{cases}$$

Note that $g'(x)$ is <u>unbounded</u>, it is <u>not</u> Riemann integrable on any closed interval $[0,\varepsilon]$ $(\varepsilon>0)$.

(2) However, if f is <u>continuously differentiable</u> on $[-\pi,\pi]$, then f' is cts on $[-\pi,\pi]$ & hence (Riemann) integrable on $[-\pi,\pi]$.

<u>Fourier Series of $2T$-periodic (real) functions</u>

Let f be a $2T$-periodic function

Then $g(x) = f(\frac{T}{\pi}x)$ is 2π-periodic

Therefore

$$f(\tfrac{T}{\pi}x) = g(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$$

(sub. $y = \frac{T}{\pi}x$)

with
$$\begin{cases} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)dx = \frac{1}{2T}\int_{-T}^{T} f(y)dy \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx dx = \frac{1}{T}\int_{-T}^{T} f(y)\cos(\frac{n\pi}{T}y)dy \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx dx = \frac{1}{T}\int_{-T}^{T} f(y)\sin(\frac{n\pi}{T}y)dy \end{cases}$$

$$f(y) \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{T} y\right) + b_n \sin\left(\frac{n\pi}{T} y\right) \right]$$

with

$$\begin{cases} a_0 = \frac{1}{2T} \int_{-T}^{T} f(y)\,dy \\ a_n = \frac{1}{T} \int_{-T}^{T} f(y) \cos\left(\frac{n\pi}{T} y\right) dy & n \geq 1 \\ b_n = \frac{1}{T} \int_{-T}^{T} f(y) \sin\left(\frac{n\pi}{T} y\right) dy \end{cases}$$

is called <u>Fourier series</u> of the <u>2T-periodic function f</u>.

§ 1.2 Riemann-Lebesgue Lemma

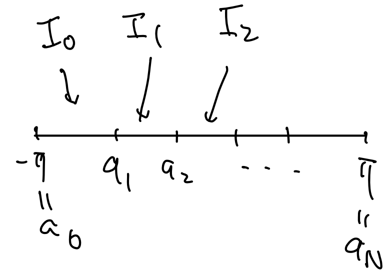
Recall: A step function on $[-\pi, \pi]$ is a function of the form

$$S(x) = \sum_{j=0}^{N-1} s_j \chi_{I_j}$$

where (i) $I_j = (a_j, a_{j+1}]$ for $j = 1, \cdots, N-1$

$I_0 = [a_0, a_1]$

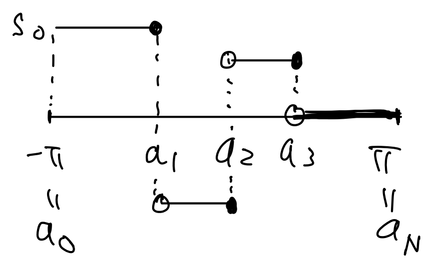
$-\pi = a_0 < a_1 < \cdots < a_{N-1} < a_N = \pi$



(ii) For a set E, $\chi_E = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$

is the characteristic function for E.

(iii) $s_j \in \mathbb{R}$, $j = 0, \cdots, N-1$.



> **Lemma 1.2** For every step function s integrable on $[-\pi, \pi]$, $\exists$ constant $C > 0$ (indep. of n, but depends on s) such that $|a_n(s)|, |b_n(s)| \leq \frac{C}{n}$, $\forall n \geq 1$
> where $a_n(s)$, $b_n(s)$ are Fourier coefficients of s.

Pf: Let $S(x) = \sum_{j=0}^{N-1} s_j \chi_{I_j}(x)$

then for $n \geq 1$

$$\pi a_n(s) = \int_{-\pi}^{\pi} \left(\sum_{j=0}^{N-1} s_j \chi_{I_j}(x) \right) \cos nx \, dx$$

$$= \sum_{j=0}^{N-1} s_j \int_{a_j}^{a_{j+1}} \cos nx \, dx$$

$$= \sum_{j=0}^{N-1} s_j \frac{1}{n} [\sin(n a_{j+1}) - \sin(n a_j)]$$

$$\Rightarrow |a_n(s)| \le \frac{1}{n} \cdot \frac{2}{\pi} \sum_{j=0}^{N-1} |s_j| = \frac{C}{n}, \quad \text{where } C = \frac{2}{\pi} \sum_{j=0}^{N-1} |s_j|$$

Similarly for $|b_n(s)| \le \frac{C}{n}$, $\forall n \ge 1$. ※

Lemma 1.3 Let f be integrable on $[-\pi, \pi]$. Then $\forall \varepsilon > 0$, $\exists$ a step function $s(x)$ such that

(i) $s \le f$ on $[-\pi, \pi]$, &

(ii) $\int_{-\pi}^{\pi} (f - s) < \varepsilon$

Pf: f (Riemann) integrable

$\Rightarrow$ f can be approximated from below by Darbour lower sums.

i.e. $\forall \varepsilon > 0$, $\exists$ partition $a_0 = -\pi < a_1 \cdots < a_N = \pi$

s.t. $$\int_{-\pi}^{\pi} f - \sum_{j=0}^{N-1} m_j (a_{j+1} - a_j) < \varepsilon$$

where $m_j = \inf\{ f(x) : x \in [a_j, a_{j+1}] \}$

Define the step function

$$s(x) = \sum_{j=0}^{N-1} m_j \chi_{I_j}(x) \qquad (\text{i.e. } s_j = m_j)$$

with $I_j = (a_j, a_{j+1}]$ for $j = 1, \cdots, N-1$

$I_0 = [a_0, a_1]$.

Then $s \le f$ & $\int_{-\pi}^{\pi} s(x)\,dx = \sum_{j=0}^{N-1} m_j (a_{j+1} - a_j)$

$\therefore \int_{-\pi}^{\pi} (f - s) < \varepsilon$. ※

Now we can prove

Thm 1.1 (Riemann-Lebesgue Lemma)
The Fourier coefficients of a 2π-periodic function f integrable on $[-\pi,\pi]$ converge to 0 as $n\to+\infty$.

Pf: $\forall\, \varepsilon>0$, Lemma 1.3 $\Rightarrow$ $\exists$ step function s s.t.

$$s\le f \quad \& \quad \int_{-\pi}^{\pi}(f-s)<\frac{\varepsilon}{2}$$

Therefore
$$|a_n(f)-a_n(s)| = \frac{1}{\pi}\left|\int_{-\pi}^{\pi}(f-s)(x)\cos nx\,dx\right|$$
$$\le \frac{1}{\pi}\int_{-\pi}^{\pi} f-s < \frac{\varepsilon}{2\pi} \qquad (\text{as } f\ge s)$$

By Lemma 1.2, $\exists\, n_0>0$ s.t.
$$|a_n(s)|<\frac{\varepsilon}{2},\ \forall\, n\ge n_0 \qquad \left(n_0=\left[\frac{2C}{\varepsilon}\right]+1, \text{ where } C \text{ as in Lemma 1.2}\right)$$

Hence
$$|a_n(f)| \le |a_n(s)| + |a_n(f)-a_n(s)|$$
$$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2\pi} < \varepsilon,\quad \forall\, n\ge n_0$$

$\therefore\ a_n(f)\to 0$ as $n\to+\infty$.

Similarly for $b_n(f)$. ※