Answers to selected exercise in tutorial 5-7.

Tutorial 5

Q2: rework Q4 of test 1, try it on case of cubic.

Sol:

Roughwork: want $(a+\frac{1}{m})^3 < c < (b-\frac{1}{m})^3$.

$$a^3+\frac{3a^2}{m}+\frac{3a}{m^2}+\frac{1}{m^3} < c.$$

prove stronger $\wedge$

$$a^3+\frac{3a^2}{m}+\frac{3a}{m}+\frac{1}{m} < c.$$

$$\rightsquigarrow \frac{1}{m}(3a^2+3a+1) < c-a^3.$$

hence take by AP $m > \frac{3a^2+3a+1}{c-a^3}$. & $m>1$.

want $(b-\frac{1}{m})^3 > c$.

$$b^3-\frac{3b^2}{m}+\frac{3b}{m^2}-\frac{1}{m^3} > c. \longrightarrow b^3-c > \frac{3b^2}{m}-\frac{3b}{m^2}+\frac{1}{m^3}.$$

prove stronger.

$$b^3-c > \frac{3b^2}{m}+\frac{1}{m} \quad \left(> \frac{3b^2}{m}-\frac{3b}{m^2}+\frac{1}{m^3}\right).$$

hence take $m > \frac{(3b^2+1)}{b^3-c}$.

By Archimedean principle, take m s.t. $\frac{1}{m} < b$, $m > \frac{3b^2+1}{b^3-c}$, $m > \frac{3a^2+3a+1}{c-a^3}$. & $m>1$.

then.

$$m > \frac{3a^2+3a+1}{c-a^3}$$

$$\frac{1}{m}(3a^2+3a+1) < c-a^3.$$

$$a^3+\frac{3a^2}{m}+\frac{3a}{m}+\frac{1}{m} < c.$$

$$\vee$$

$$a^3+\frac{3a^2}{m}+\frac{3a}{m^2}+\frac{1}{m^3}.$$

hence $(a+\frac{1}{m})^3 < c$.

similarly. $m > \frac{3b^2+1}{b^3-c}$.

$$b^3-c > \frac{3b^2}{m}+\frac{1}{m} > \frac{3b^2}{m}-\frac{3b}{m^2}+\frac{1}{m^3}.$$

hence $b - \frac{3b^2}{m}+\frac{3b}{m^2}-\frac{1}{m^3} > c$.

$$(b-\frac{1}{m})^3 > c$$

$\blacksquare$

Q4: section 3.4.19 : show if $(x_n), (y_n)$ are bounded sequences, then $\limsup(x_n+y_n) \le \limsup(x_n) + \limsup(y_n)$.

Sol.

Ver 1. By hw 3B, we know $\exists$ subsequence of limit $\to$ limsup. and also such limit is maximal choice.

now take a subsequence of sequence $X+Y = (x_n+y_n)$, that converge to $\limsup(X+Y)$, say denoted l.

now take a subsequence of this subsequence of X-part, its maximal possible limit is bounded above by $\limsup X$. say the limit is l_1.

And the corresponding subsequence on Y converge to l_2.

hence $l = l_1 + l_2$ by limit thm.

but l_1 is just a possible subsequence limit of X hence $l_1 \le \limsup X$.

Similarly $l_2 \le \limsup Y$.

hence $\limsup(X+Y) = l = l_1 + l_2 \le \limsup X + \limsup Y$.

Ver.2. proof by definition:

denote $l := \limsup(x_n+y_n) = \limsup(X+Y)$.

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \ge N$, $\left|\left(\sup_{k\ge n}(x_k+y_k)\right) - l\right| < \varepsilon$.

hence $\sup_{k\ge n}(x_k+y_k) + \varepsilon > l$.

by seeing $\{x_k+y_k\}_{k\ge n}$ as subset of $\{x_k+y_i\}_{k,i\ge n}$.

we have $\sup_{k\ge n}(x_k) + \sup_{k\ge n}(y_k) + \varepsilon$

$\ge \sup_{k\ge n}(x_k+y_k) + \varepsilon > l$.

hence by limiting process, $\varepsilon \to 0$,

$\limsup X + \limsup Y \ge l = \limsup(X+Y)$.

$\blacksquare$

Tutorial 6

Q2. if (a_n) convergent & (b_n) bounded sequence,
then $\limsup(a_n+b_n) = \limsup a_n + \limsup b_n$.

Ans: $\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$, $\left| \left[\sup_{k \geq n} (x_k + y_k) \right] - l \right| < \varepsilon$.

also $\exists N_2 \in \mathbb{N}$ s.t. $\forall n \geq N_2$,
denote $l := \limsup (A+B)$,
$r := \limsup (A)$.

hence for $k \geq n$,

$$r - \varepsilon < x_k < r + \varepsilon$$

$$r - \varepsilon + y_k < x_k + y_k < r + \varepsilon + y_k$$

$$r - \varepsilon + \sup_{k \geq n} y_k < \sup_{k \geq n} (x_k + y_k) < r + \varepsilon + \sup_{k \geq n} y_k.$$

hence $\left| \left(r + \sup_{k \geq n} y_k \right) - l \right| < 2\varepsilon$. $\forall n \geq N_1, N_2$.

hence $r + \sup_{k \geq n} y_k \to l$.

but $r + \sup_{k \geq n} y_k \to r + \limsup B$

hence $\limsup A + \limsup B = l = \limsup (A+B)$.

$\blacksquare$