MATH2050B
Tutorial Solution

November 27, 2022

Abstract

In this note, solution of many problems of each tutorial are provided. Some are detailed while some are just provided with numerical answer since similar problem have been solved here or in previous solution. So this note should be read with the following documents: 1. homework solution available on balckboard, 2. solutions to tests available on blackboard, 3. solutions to previous tutorials available on course webpage.

Tutorial 1

Note solutions to tutorial 1 is entirely provided in previous tutorial solution.

Tutorial 2

Section 2.3

1 Note it is of similar type as below solution, so we do not write out details of proof. Hence It has lower bounded 0 and infimum exists equal to 0. No upper bound, hence supremum also doesn't exist.

2 Note that its solution is entirely provided in previous tutorial solution.

14 Let X be set bounded below.

Forward Direction: Suppose a lower bound w is the infimum of S. Then since s is the greatest upon all lower bound, any number strictly larger than w is not an lower bound, i.e. for any $\epsilon > 0$, $w + \epsilon$ is not a lower bound. Hence there exists $t \in S$ greater than $w + \epsilon$, i.e. $t < w + \epsilon$.

Backward Direction: Suppose for any $\epsilon > 0$, there exists $t \in S$ such that $t < w + \epsilon$. Hence any number strictly large than w is not a lower bound. Hence any possible lowe bound is smaller than or equal to w, hence w is the greatest among all lower bound.

Section 2.4

1 Note that the set is bounded above by 1. And for any $\epsilon > 0$, take $a = 1 - \epsilon$. Now take $n > \frac{1}{\epsilon}$, hence as an element of the set, $1 - \frac{1}{n} > 1 - \epsilon$, hence a cannot be an upper bound. Hence 1 is the supremum.

2 Note that its solution resemble above, the infimum is then -1 and supremum is 1.

11 For any $x \in X, y \in Y$,

$$h(x, y) \leq f(x)$$

by varying y, fixing x. Similarly by varying x, fixing y, we have

$$h(x, y) \geq g(y)$$

Hence collectively,

$$g(y) \leq h(x, y) \leq f(x)$$

But note that the left side is independent of x, hence you can just take supremum and retain the inequality. Then similarly you take infimum on the right.

$$\sup g(Y) \leq f(x)$$

$$\sup g(Y) \leq \inf f(X)$$

12 This solution is already provided in previous solution.

16 See the note for possible solution to test 1, Q3. That solution provides a way to find a_n, b_n that is of diminishing distance but convergent to the square root.

Section 2.5

7 Note it is of similar type as below solution.

8 This solution is already provided in previous solution.

9 Note it is of similar type as below solution.

Section 3.1

12 This solution is already provided in previous solution.

16 Note it is of similar type as above solution.

17 Though it is of similar type as above solution, we work on it. We make use of the fact that the geometric sequence for ratio $0 < r < 1$ is convergent to 0.

For any $\epsilon > 0$, take by above fact some $N \in \mathbb{N}$ such that $(\frac{2}{3})^N < \frac{\epsilon}{9}$, then for any $n \geq N$,

$$\begin{aligned}
|\frac{2^n}{n!} - 0| &= \frac{2^n}{n!} \\
&< 2 \cdot (\frac{2}{3})^{n-2} \\
&< 4.5 \cdot \frac{\epsilon}{9} \\
&< \epsilon
\end{aligned}$$

Hence is convergent to 0.

18 Let $\epsilon = \frac{x}{2}$, by definition of convergence, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $|x_n - x| < \epsilon = \frac{x}{2}$.

Hence

$$\frac{x}{2} < x_n < \frac{3x}{2} < 2x$$

Section 3.2

2 For part a, take $X = ((-1)^n)_n$, $Y = ((-1)^{n+1})_n$. Then both are divergent, but their sum is constant zero sequence, hence convergent. For part b, can use same sequence, then product is constant -1 sequence, hence convergent.

3 Note by theorem in the section, if two sequence are convergent, then their difference are also convergent. Take $Z = X + Y$ sequence, hence by condition, it is convergent. Hence Y appear as difference of two convergent sequences Z, X, hence it is convergent.

6 (a) This is just sum of sequences $(4)_n + (\frac{4}{n})_n + (\frac{1}{n^2})_n$, hence by limit theorem is of limit $4 + 0 + 0 = 4$.

(b) For any $\epsilon > 0$, take $N \in \mathbb{N}$ such that for any $n \geq N$, $\frac{1}{n} < \epsilon$. Now

$$|\frac{(-1)^n}{n+2} - 0| = \frac{1}{n+2} < \frac{1}{n} < \epsilon$$

(c) Note that the sequence can be rewritten as

$$(\frac{1 - \frac{1}{\sqrt{n}}}{1 + \frac{1}{\sqrt{n}}})$$

Both sequence on the numerator and denominator are convergent with nonzero limit 1, hence by limit theorem, the limit is their quotient hence 1.

(d) This sequence can be rewritten as sum of sequences

$$(\frac{1}{\sqrt{n}})_n + (\frac{1}{n})_n \cdot (\frac{1}{\sqrt{n}})_n$$

Sequence involved are just all convergent to 0, hence by limit theorem, the total limit is $0 + 0 \cdot 0 = 0$.

19 The solution of all these are just similar to above question in section 3.1. We leave out details for you to fill in.

(a) Convergent to 0.

(b) Properly divergent to ∞.

(c) Convergent to 0.

(d) Convergent to 0, just see it as product of $\frac{1}{n} \cdot \frac{n!/1}{n^{n-1}}$ which the latter factor is strictly < 1.

Tutorial 3

Homework 2

4 This is already provided in previous solution, or even in possible solution to test1.

10 This is provided in homework solution.

Section 2.4

11 This solution is provided in corresponding problem above in tutorial 2.

12 This solution is provided in corresponding problem above in tutorial 2.

19 Take n by Archimedean principle such that $\frac{u}{n} < (y-x)$. Take by completeness property (on set $\{m : \frac{mu}{n} \leq x\}$) the supremum m, hence $\frac{mu}{n} \leq x$ and by its nature as supremum, $\frac{(m+1)u}{n} > x$ and also by choice of n we have $\frac{(m+1)u}{n} < y$. Hence $r = \frac{m+1}{n}$.

Section 3.1

6 (a) For any $\epsilon > 0$, by Archimedean principle, there exists $N \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon^2$. Hence for any $n \geq N$,

$$|\frac{1}{\sqrt{n+7}} - 0| < \frac{1}{\sqrt{n}} < \epsilon$$

Hence convergent to 0.

(b) For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{4}$. Hence for any $n \geq N$, we have

$$|\frac{2n}{n+2} - 2| = \frac{4}{n+2} < \frac{4}{N} < \epsilon$$

Hence convergent to 2.

(c) For any $\epsilon > 0$, by Archimedean principle, there exists $N \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon^2$. Hence for any $n \geq N$,

$$|\frac{\sqrt{n}}{n+1} - 0| = \frac{1}{\sqrt{n} + \frac{1}{\sqrt{n}}} < \frac{1}{\sqrt{n}} < \epsilon$$

(d) For any $\epsilon > 0$, by Archimedean principle, there exists $N \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Hence for any $n \geq N$,

$$|\frac{(-1)^n n}{n^2+1} - 0| = \frac{1}{n + \frac{1}{n}} < \frac{1}{n} < \epsilon$$

8 For forward direction, for any $\epsilon > 0$, there exists N such that for any $n \geq N$ that

$$||x_n| - 0| \leq |x_n - 0| < \epsilon$$

Hence convergent to 0.

For backward direction, for any $\epsilon > 0$, there exists N such that for any $n \geq N$ that

$$|x_n - 0| = ||x_n| - 0| < \epsilon$$

Hence convergent to 0.

18 This is already done in above.

Section 3.2

5 (a) For any $\ell > 0$, take by Archimedean principle N such that $2^{N'} > |\ell| + 1$. Take $\epsilon = 1$, for any $N \in \mathbb{N}$, take $n = \max N', N$, then

$$|2^n - \ell| = 1 = \epsilon$$

Hence cannot have ℓ as limit. Hence cannot have limit.

(b) As above but now take $n^2 > |\ell| + 1$. Solution resemble above.

6 This is done above.

11 This problem is easy to solve if one is allowed to use logarithm, we present instead one solution without using log for part a. Part b is just by definition. Note that part a is also done in previous tutorial solution.

This sequence is multiple of two sequences $(3^{1/2n})_n$ and $(\sqrt{n}^{1/2n})_n$. The first sequence converge to 1 obviously, the second is what to be proved.

For any $\epsilon > 0$, take $N > 0$ such that $\frac{1}{N} < \epsilon^2, N > 2$. Hence we have for any $n \geq N$,

$$\begin{aligned}
n &< 1 + 4n < 1 + 4n(n\epsilon^2) \\
n &< 1 + 4n(n\epsilon^2) < 1 + 4n\epsilon + 4n(2n-1)\epsilon^2 < (1+2\epsilon)^{2n} \\
n^{1/2n} &< 1 + 2\epsilon < 1 + 2\epsilon + \epsilon^2 \\
\left|(\sqrt{n}^{1/2n}) - 1\right| &< \epsilon
\end{aligned}$$

Hence the point is to do roughwork so that you can take appropriate N here then reverse all roughwork step you have written.

19 This is done above.

Section 3.3

1 This sequence is bounded below by 4 (shown inductively $x_{n+1} = \frac{1}{2}x_n + 2 > 2 + 2 = 4$ and bounded above by 8, which is to be proven inductively that $x_{n+1} = x_n + 2 \le \frac{8}{2} + 2 = 6 < 8$.

Next we show it is decreasing since $x_n - x_{n+1} = \frac{1}{2}x_n - 2 > 2 - 2 = 0$ inductively.

Now by monotone convergence theorem, it has a limit say x. Now $(x_{n+1})_n = (\frac{1}{2}x_n + 2)$ the 1-tail subsequence has same limit as original convergent subsequence. But from its expression and limit theorem, it also have limit as $\frac{x}{2} + 2$. Hence we get equation $x = \frac{x}{2} + 2$, hence $x = 4$ is the limit.

2 This is done in previous tutorial solution.

3 This is just involving longer arithmetic but nothing different.

Section 3.4

1 Merge the two sequences $(1)_n, (n)_n$ such that the odd index term from first sequence while even index from second sequence.

4 (a) Suppose there exists ℓ that is limit of the sequence, take $\epsilon = \frac{1}{4}$, there exists $N > 0$ such that for any $n \ge N$ and $n \ge 2$, we have $|(1 - (-1)^n + \frac{1}{n}) - \ell| < \epsilon = \frac{1}{4}$. Now suppose WLOG one such n is odd, $|2 + \frac{1}{n} - \ell| < \frac{1}{4}$, hence $\ell > 2 + \frac{1}{n} - \frac{1}{4} > 1$. For $n+1$, we have $|\frac{1}{n+1} - \ell| < \frac{1}{4}$. Hence $\ell < \frac{1}{n+1} + \frac{1}{4} < \frac{1}{2}$. Hence contradiction.

(b) Similar to above.

Section 3.5

3 (a) Take $\epsilon = 1$, then for any $N > 0$, take $m = N, n = N + 1$, then $|x_m - x_n| = 2 \ge \epsilon = 1$. Hence not Cauchy.

(b) Take $\epsilon = \frac{1}{2}$, then for any $N > 0$, take $n > N$ and $n > 4$, similarly $m = n + 1$, then $|x_m - x_n| > |1 - \frac{1}{n} - \frac{1}{m}| \ge \frac{1}{2} = \epsilon$. Hence not Cauchy.

(c) Take $\epsilon = 1$, then for any $N > 0$, take $m = 10^N, n = 10^{N+1}$, then $|x_n - x_m| \ge 2 \ge 1 = \epsilon$. Hence not Cauchy.

Tutorial 4

Section 3.2

4 Note that if X converges to nonzero limit, there exists a K-tail which X is bounded away strictly from 0. By limit theorem, the K-tail of Y appear as quotient well-defined convergent K-tail sequence of (XY) and (X) of which denominator has nonzero limit. Hence K-tail of Y is convergent, hence Y is convergent.

5 This is done above.

6 This is done above.

22 Yes, it is convergent. For any $\epsilon > 0$, there exists $N' > 0$ such that for any $n \geq N'$, $|x_n - \ell_x| < \frac{\epsilon}{2}$. Similarly there exists $M > 0$ such that for any $n \geq M$, $|x_n - y_n| < \frac{\epsilon}{2}$.

Now take $N = \max N', M$, then for any $n \geq N$, $|y_n - \ell_x| \leq |y_n - x_n| + |x_n - \ell_x| < \epsilon$.

Section 3.3

This is all done above.

Section 3.4

12 Since it is unbounded, we assume it is unbounded above WLOG. There exists term x_{k_1} that $x_{k_1} > 1$. Now by unboundedness, take x_{k_2} such that it is larger than $M = \max\{x_1, x_2, \cdots, x_{k_1}\} + 1$ (note it is ≥ 2). Now inductively choose $x_{k_{i+1}}$ such that it is larger than all terms before and including x_{k_i} and plus 1, inductively it is $\geq i+1$. Hence this subsequence $(x_{k_i})_i$ is unbounded properly that $(1/x_{k_i})_i$ is bounded above by sequence $(\frac{1}{i})_i$, hence convergent to 0.

13 Please refer to textbook to construct on your own.

17 The limit superior is first by taking supremum of tails, then take limit. The supremum sequence is $(22, \frac{3}{2}, \frac{3}{2}, \cdots, ...)$ such that each term $(1 + \frac{1}{n}$ occur twice (for $n > 1$). Hence its limit is 1.

For limit inferior, the infirmum sequence is $(-1, -1, -\frac{1}{2}, -\frac{1}{2}, \cdots)$ such that each $-\frac{1}{n}$ occur twice, hence limit is 0.

18 For the backward direction, suppose limit sup and limit inf are equal, denoted ℓ. Then the two sequences responsible for them are $(y_n)_n, (z_n)_n$ which y_n, z_n are supremum/infimum of n-tail resp.. Hence naturally $y_n \geq x_n \geq z_n$ such that the bounding sequence converge to limit ℓ, hence by squeeze theorem, x_n is convergent to smae limit.

Conversely, suppose $(x_n)_n$ is convergent to ℓ. For any $\epsilon > 0$, there exists $N > 0$ such that for any $n \geq N$, $|x_n - \ell| < \frac{\epsilon}{2}$. Hence $\ell - \frac{\epsilon}{2} < x_n < \ell + \frac{\epsilon}{2}$. Hence with same notation $(y_n)_n, (z_n)_n$ as above. We have $\ell - \frac{\epsilon}{2} \leq z_n \leq y_n \leq \ell + \frac{\epsilon}{2}$ for any $n \geq N$. Hence $|y_n - \ell| < \epsilon$, similarly for z_n. Hence the limit sup and inf equal ℓ and hence are equal.

19 Let $(a_n), (b_n), (c_n)$ be the sequence of n-tail supremum of $(x_n), (y_n), (x_n + y_n)$ respectively. They are responsible for limsup of each sequence resp.. Now the set $\{x_k + y_k\}_{k \geq n}$ is a subset of $\{x_k\}_{k\geq n} + \{y_k\}_{k \geq n} := \{x_k + y_i\}_{i,k\geq n}$, where the plus sign is for elementwise addition. Now the supremum of first set is c_n, while supremum of latter set is just $b_n + c_n$ (from boundedness and additivity of supremum of sum of set). Hence by subset relation, $c_n \leq a_n + b_n$. Going to limit, hence we have the desired inequality.

One occassion they are unequal is to consider sequence $(2, 0, 2, 0, \cdots), (0, 1, 0, 1, \cdot)$. The limsup of them are $2, 1$, sum to 3. But sum of sequence is just $(2, 1, 2, 1, \cdots)$ hence of limsup 2, hence not equal.

Section 3.5

1 Consider sequence $((-1)^n)_n$, details are left to you.

2 (a) For any $\epsilon > 0$, take N such that $\frac{1}{N} < \frac{\epsilon}{2}$, for any $m, n \geq N$, we have $|x_m - x_n| = |\frac{1}{m} - \frac{1}{n}| \leq \frac{2}{N} < \epsilon$, hence Cauchy.

(b) For any $\epsilon > 0$, take N such that $\frac{1}{2^{N-1}} < \epsilon$, for any $m, n \geq N$, assume WLOG $m \geq n$, we have $|x_m - x_n| = \frac{1}{(n+1)!} + \cdots \frac{1}{m!} \leq \frac{1}{2^N} + \cdots + \frac{1}{2^{m-1}} < \frac{1}{2^{N-1}} < \epsilon$, hence Cauchy.

3 (a) Take $\epsilon = 1$, for any $N > 0$, take $n = N, m = N + 1$, hence $|x_m - x_n| = 2 > \epsilon = 1$. Hence not Cauchy.

(b) Take $\epsilon = \frac{1}{2}$, for any $N > 0$, take $n = N + 4, m = N + 3$, hence $|x_m - x_n| \geq 1 - \frac{2}{N+4} > \epsilon = \frac{1}{2}$. Hence not Cauchy.

(c) Take $\epsilon = 1$, for any $N > 0$, take $n = 2^N, m = 2^{N+1}$, hence $|x_m - x_n| = 1 \geq \epsilon = 1$. Hence not Cauchy.

12

$$
\begin{aligned}
|x_{n+1} - x_n| &= |\frac{1}{2 + x_n} - \frac{1}{2 + x_{n-1}}| \\
&= |\frac{x_n - x_{n-1}}{(2 + x_n)(2 + x_{n-1})}| \\
&\leq \frac{1}{4}|x_n - x_{n-1}|
\end{aligned}
$$

Here we make use of fact that all terms are nonnegative, which is easy to show. Its contractive constant is $\frac{1}{4}$. Now let its limit be x, then we have equation $x = (2 + x)^{-1}$, hence the only possible choice is $-1 + \sqrt{2}$.

13 This is similar to above except to note that $x_n \geq 2$ for any n. Obiouvsly contractive constant is $\frac{1}{4}$ and limit is $1 + \sqrt{2}$.

Tutorial 5

Note that the tutorial exercise should be on section 3.4 but not on 3.5.

Auxiliary Questions

2 This is provided in previous solution.

3 This is already provided in possible solution to test 1.

Section 3.4

12 This is done above.

13 This is done above.

19 This is done above.

Tutorial 6

Auxiliary Questions

1. Here we just provide the solution, details are to be written out by yourself. Note in particular that some sequence are not bounded, hence at least one of limsup or liminf cannot be defined.

   (a) It is convergent to 0, hence limsup and liminf are 0.

   (b) It is properly divergent, hence no limsup or liminf.

   (c) It has convergent to 0, hence limsup and liminf are 0.

   (d) It has limsup 2, liminf 0.

   (e) It is convergent to 2, hence limsup and liminf are 2.

   (f) It has limsup 1, liminf -1.

   (g) It has limsup 3, liminf -1.

   (h) It has limsup 1, liminf -1. (Note the sequence is very complicated).

   (i) It is not bounded above or below, hence no limsup or liminf.

   (j) It is properly divergent, hence no limsup or liminf.

   (k) The author doesn't know the answer.

   (l) The author doesn't know the answer.

2. This is already provided in previous solution.

3. Since the liminf is $\ell > 1$, take ϵ such that $\ell - \epsilon = c > 1$. Now there exists $N > 0$ such that for any $n \geq N$, $\inf\{a_{k+1}/a_k\}_{k \geq n} > \ell - \epsilon = c$. Hence for all k, $a_{N+k} = (a_{N+k}/a_{N+k-1}) \cdots (a_{N+1}/a_N) > c^k a_N$. Hence this sequence can be bounded below by geoemtric sequence of raio > 1 on its tail, hence cannot be convergent. It cannot contain convergent subsequence since by above reasoning (you fill in yourself), we have the ratio of consective terms is always bounded below by some $\ell' > 1$. Hence the same condtion on liminf hold on the subsequence. Hence cannot be convergent. If $\ell = 1$, it can be convergent, just consider constant sequence.

Section 3.7

We just write out nexcessary ideas and solutions, remaining details are for you to fill in.

10 Just the question just ask to follow the example 3.7.6(f), we just write out necessary detail for you to fill in. Note that the partial sum s_n for n odd is an increasing sequence, while for n even is a decreasing sequence. And obviously bounded below by -2 and above by 0. Hence both subsequences have limit. Now if one can show their limit are equal, then the series converge. Note that two sequence just differ by a term of order $O(\frac{1}{\sqrt{n}})$, hence just of same limit, hence convergent.

11 It is convergent since $a_n \to 0$. For $\epsilon = \frac{1}{2}$, there exists N-tail of (a_n) such that it is all $< \frac{1}{2}$. Since convergence of partial sum do not need first N-terms. We see that $(a_n^2)_{n\geq N}$ is just bounded above by that N-tail. Hence by positivity and montone covnergence theorem, the square series converge.

12 No, consider sequence $(\frac{1}{n^2})_n$.

13 Yes, this sequence by inequality is bounded by $\frac{1}{2}(a_n^2+a_{n+1}^2)$, which by previous question is summable, i.e. series converge.

14 This sequence is bounded below by sequence $(\frac{a_0}{n})_n$, which is just a sclar multiple of harmonic sequence, hence is not summable, i.e. sum diverges.

15 The inequality is easy to show, group in two ways, the first is $(1), (2), (3,4), (5,6,7,8), \cdots, (2^{n-1}+1, \cdots, 2^n)$, hence for each term, they bounds below the sum $\frac{1}{2}a_1, a_2, 2a_4, \cdots, 2^{n-1}a_{2^n}$. The second way is $(1), (2,3), (4,5,6,7), \cdots, (2^{n-1}, \cdots, 2^n - 1), (2^n)$, hence bounded above by $a_1, 2a_2, 4a_4, \cdots, 2^{n-1}a_{2^{n-1}}, a_{2^n}$, hence leading to the inequality.

Now suppose the partial sum of a_n converges, then $a_n \to 0$. Now by the inequality, we know this sequence bounds below the half the partial sum of $2^n a_n$, hence the latter partial sum should converge, hence summable.

Conversely, if the partial sum converges, then this partial sum plus a term (which tends to 0) bounds the partial sum of a_n, hence partial sum of a_n should converges.

16 Use Cauchy condensation test, then the partial sum for $2^n a_n$ becomes partial sum of $\frac{1}{2^{n(p-1)}}$. Note that this is a geoemtric series, convergent iff ratio is $< 1, > 0$, hence we need $p > 1$. Else it is divergent.

17 (a) Use Cauchy COndensation test, this become sum of $\frac{1}{n}$, hence divergent.

(b) Use Cauchy condensation test, become $\frac{1}{n \log n}$, hence by above is divergent.

(c) Use the condensation test, become sum of $\frac{1}{n \log n \log\log n}$, hence divergent.

18 (a) Use Cauchy condensation test, this become sum of $\frac{1}{n^c}$, hence convergent sum if $c > 1$.

(b) Use Cauchy condensation test, this become form in part a, hence convergent.

Section 4.1

9 (a) We work only two problem there. For any $\epsilon > 0$, take $\delta > \frac{1}{2}, \epsilon/4$, hence for any y such that $|y-2| < \delta$, we have $|\frac{1}{1-y} - (-1)| = \frac{|2-y|}{y-1} < 2\delta < \epsilon$, hence convergent to that limit.

(b) Similar as above.

(c) For any $\epsilon > 0$, take $\delta > \epsilon/2$, hence for any y such that $|y - 0| < \delta$, we have $|\frac{y^2}{|y|} - 0| = |y| < \epsilon$, hence convergent to that limit.

(d) Similar as above.

10 (a) This is similar to solution below and above.

(b) For any $\epsilon > 0$, take $\delta > \epsilon/7$, hence for any y such that $|x+1| < \delta$, we have $|\frac{x+5}{2x+3} - 4| = \frac{7|x+1|}{2x+3} < 7|x+1| < \epsilon$, hence convergent to that limit.

11 Same as above.

12 (a) Suppose its limit exist and equal to ℓ. Take $\epsilon = 1$, for any $\delta > 0$, take $y = \frac{1}{N}$ such that $\frac{1}{N} < \delta$ and $N > |\ell| + 1$. then $|\frac{1}{y^2} - \ell| \geq 1$. Hence no limit there.

(b) Same as above.

(c) Suppose its limit exist and equal to ℓ. Take $\epsilon = \frac{1}{2}$, for any $\delta > 0$, take $y = \frac{1}{N}, y' = -\frac{1}{N}$ such that $\frac{1}{N} < \delta, N > 4$. Now if $|(y + \text{sgn } y) - \ell| = |1 + \frac{1}{N} - \ell| < \epsilon = \frac{1}{2}$, $|(y' + \text{sgn } y') - \ell| = |1 + \ell + \frac{1}{N}| < \epsilon = \frac{1}{2}$. But then $2 + \frac{2}{N} \leq |\cdots| + |\cdots| < \frac{1}{2} + \frac{1}{2} = 1$, contradiction. Hence no limit there.

(d) This is more complicated but similar idea.

Tutorial 7

Section 4.1

Done before

Section 4.2

For this session, we encourage you to read more examples from the book. Therefore some solutions are very short.

1. Similar as section 4.1, but instead use algebraic rules.
2. (a) First use quotient rule on the quotient, then use product rule for the square root.
   (b) This is same as limit of $x+2$.
   (c) This is same as limit of $x+2$.
   (d) This is same as limit of $\frac{1}{\sqrt{x}+1|}$.
3. This is same as limit of $\frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2} \cdot \frac{\sqrt{1+2x}+\sqrt{1+3x}}{\sqrt{1+2x}+\sqrt{1+3x}} = -\frac{1}{1+2x} \cdot \frac{1}{\sqrt{1+2x}+\sqrt{1+3x}}$.
4. $\cos(\frac{1}{x})$ bounded between $-1, 1$. Hence the second limit follows from squeeze theorem.

Section 4.3

4 Suppose limit of f at c is ∞. for any $\epsilon > 0$, take N such that $\frac{1}{N} < \epsilon$, hence for this $N > 0$, there exists $\delta > 0$ such that for any $y \in V_\delta(c)$, $f(y) > N$. Hence in this neighbourhood, $0 < \frac{1}{f(y)} < \frac{1}{N}$, hence $|\frac{1}{f(y)} - 0| < \frac{1}{N} < \epsilon$, hence of limit 0.

COnversely, for any $N > 0$, take $\epsilon = \frac{1}{N}$, for this $\epsilon > 0$, there exists $\delta > 0$ such tha tfor ay $y \in V_\delta(c)$, $|\frac{1}{f(y)} - 0| < \epsilon = \frac{1}{N}$, hence $f(y) > N$ in this neighbourhood, hence convergent to ∞.

5 (a) Not exist. We work only over one problem here. Suppose its limit exist and equal ℓ. Take $\epsilon = 1$, for any $\delta > 0$, take $N > 0$ such that $\frac{1}{N} < \delta$ and $N > |\ell| + 1$, take $y = 1 + \frac{1}{N} \in V_\delta(1)$, hence $|\frac{y}{y-1} - \ell| \geq N - |\ell| > 1 = \epsilon$. Hence not exist.
   (b) Not exist.
   (c) Not exist.
   (d) Not exist.
   (e) Not exist.
   (f) Exist and $= 0$.
   (g) Exist and $= 1$.
   (h) Exist and $= 1$.

Tutorial 8

Auxiliary Questions

Note that some details are required to fill in yourself.

1. Either use $\epsilon - \delta$ terminology, or just use the function $f(x) = x$ to build up polynomial and show polynomial are hence continuous by limit theorems. Then use quotient theorem to show their quotient are continuous wenever they are defined.

2. Suppose x is rational, then there exists an approximating sequence of distinct rational numbers $(r_i)_i \to x$. But note that r_i in simplest form must have increasing size of denominator, hence $f(r_i) \to 0$, but $f(x) \neq 0$, hence discontinuous.

   Suppose x is irrational, with above reasoning and sequential criterion, any sequence tending to x either have a subsequence of rational numbers, which by above reasoning, its functional value tends to 0. Or also of subsequence of irrational numbers, hence functional value is 0. Hence is continuous there by sequential criterion.

3. There is no such function, check it out on math stackexchange. NOte that set of continuity must be G_δ set, which $\mathbb{Q}$ is not.

4. This is done above.

Tutorial 9

Auxiliary Questions

1. This is just by limit theorems.

2. Suppose a continuous function's value on dense set D is known. Then for any x, there exists a sequence of elements from D which tend to x, hence $f(x)$ is limit of functional value of such sequence, hence well-defined by continuity. Hence the sequence is uniquely determined.

3. We consider the dense set $D_2 = \{\frac{m}{2^n} : n \in \mathbb{N}, m \in \mathbb{Z}\}$ the set of dyadic numbers. This is dense (prove it yourself). Now value of additive function on D_2 is known, because firstly if $f(x)$ is known, $f(mx) = f(x) + \cdots + f(x) = mf(x)$ is known. Secondly, if $f(x)$ is known, then $f(x) = f(x/2^n) + \cdots + f(x/2^n) = 2^n f(x)$, hence $f(x/2^n) = f(x)/2^n$. Now note that if we fix $f(1) = c$, then by above two principles, f is known on set of dyadic numbers, hence known on whole real set.

4. Use compactness, or prove in manner like proving uniform continuity. Suppose it is not uniformly bounded, we have a bounded sequence on the interval $(x_i)_i$ such that $f(x_i) \geq i$, hence by Bolzano Weiestrass theorem, take convergent subsequence of $(x_i)_i$ as $(x_{i_k})_k \to x$. But note that $f(x)$ is locally bounded, i.e. there exists bound M such that $|f(y)| < M$ in neighbourhood of x. But a tail of that convergent subsequence will be inside this neighbourhood. But they are unbounded, contradiction. Hence should be uniformly bounded.

5. Use characterization of intervals, suppose $a = f(x), b = f(y) \in f(I)$ and WLOG $a < b$. let J be the interval of boundary point a, b, then by intermediate value theorem, for any $c \in J$, there exists z in the interval of x, y that $f(z) = c$. Hence $J \subset f(I)$, hence is an interval by characterization of interval.

6. Hence $f(\mathbb{R})$ is an interval, but there is no interval (except constant one) that contains no rational number, hence should be constant.

7. This is wrong since one can find function such that f is continuous/uniformly continuous but locally not of order $O(x)$.

8. SImilar as above.

Tutorial 10

Section 5.1

5 By definition of continuity, if we can define f at $x = 2$ as its limit, then it is continuous. But the limit of such function is just limit of $x + 3$, hence $f(2) := 5$ is continuous.

8 Since f is continuous, $(x_n) \to x$ such that each term are element in zero set, then $0 = f(x_n)$ tend to $f(x)$ by continuity, hence $f(x) = 0$, hence $x \in S$.

13 As in similar problem in tutorial 9, this is continuous then limit form rational part and irrational part agrees, hence $2x = x + 3$, hence a possible choice is 3. Now we check that it is continuous at 3, which is easy.

Section 5.2

3 Consider $f(x) = 1$ on ratioanl while $= -1$ on irrational, it is discontinuous at $c = 0$, but let $g = -f$ also discontinuous there, then $f + g = 0$ is continuous at 0. Similarly $fg = -1$ is constant continuous there.

12 For any y, since $f(y) = f(x_0) + f(y - x_0)$ and for any z near y, $f(z) = f(x_0 + (z - y)) + f(y - x_0)$, the second term is constant while first term is just function at x_0 translated by $z - y$, hence continuous when $z - y = 0$. Hence f is continuous at y, hence everywhere.

14 Note $g(0) = 1$ or 0. Suppose g is continuous at 0, note that for any y, $g(y) = g(y)g(0)$, and for any z near y, $g(z) = g(z - y)g(y)$, which second factor is constant and first factor is just g translated by y at 0, hence still continuous at 0. Hence g is continuous at y and hence everywhere.

15 The formula shows that one can write h as combination of continuous functions, hence continuous. THe checking of formula is to be doen by you.

Section 5.3

1 By extreme value theorem, there exists a minimum m such that $m = f(x_0) \leq f(x)$ for x on the interval. But $m = f(x_0) > 0$, hence take $\alpha = m$.

13 This is to show one can use uniformity from different cases collectively.

19 Try $g(x) = \frac{1}{x}$ on $(0, 1)$.

Section 5.4

7 They are obviously Lipchitsz, hence uniformly continuous. The product is not uniformly continuous by taking $\epsilon = 1$, for each $\delta_i = \frac{2}{i}$, take $x_i = 2i\pi, y_i = 2i\pi + \frac{1}{i}$ (note we need i such that $\sin x \geq \frac{1}{2x}$ on interval $[0, \frac{1}{i}]$). Now $|x_i \sin x_i - y_i \sin y_i| =\geq 2i\pi \sin \frac{1}{i} > \pi > 1 = \epsilon$. Hence not uniformly continuous.

8 For any ϵ, tak δ_ϵ for f to be ϵ-uniformly continuous. Then for this δ_ϵ, take δ' such that g is δ_ϵ-uniformly continuous. Then on δ-neighbourhood, f is ϵ-uniformly continuous.

13 This is to show uniformity preserves if function are "uniformly close", try it yourself.

14 Find δ such that f is ϵ-uniformly continuous on interval $[0, 2p]$. Then by periodicity, this δ suffices on whole real line.

Section 5.5

1 This follows from definition.

3 Both function are obviously strictly increasing, but the product is quadratic having only two roots $0, 1$, if it is monotone, then it should be constant between two roots, but they are not, hence not monotone.

5 Use sequential criterion and monotone convergence theorem for sequences.

8 If for contradiction that $f^{-1}(y) \geq g^{-1}(y)$, then we have $y = f(f^{-1}(y)) \geq f(g^{-1}(y)) > g(g^{-1}(y)) = y$, contradiction.

Tutorial 11

Auxiliary Question

1. For the first question, it is yes by considering constant function. For the second, it is no if the closed interval is bounded (hence compact), because it maps interval to interval and also functions have minimum and maximum on closed and bounded intervals. In the case the interval is noncompact, it is yes by taking a half-infinite open sub-interval/ray and then map it to whole real line $\mathbb{R}$, hence it maps to open interval which is $\mathbb{R}$.

2. No, take any example in usual calculus course whose linearization at point $x = 0$ is the vertical axis, then it cannot be Lipschitz. There are example of such function which is simutaneously monotone, hence cannot be Lipschitz.

3. No, if it has discontinuity somewhere, there is a gap given by difference of right and left limit at that point, hence not surjective in this range, contradiction.

4. Use a filtration by D_n consisting of all points of gaps (defined above) $\geq \frac{1}{n}$. Hence this can at most be a finite set. Any discontinuity must imply a gap, hence included in such filtration, hence set of discontinuity is just union of all such set, hence at most countable. Note however such set can be dense, you can consider the function f_n which is monotone but discontinuous only on the rational point r_n. By letting all these r_n exhausts the whole rational number set and making f_n summing to finite number, it can be shown that their partial sum converge to a function which is monotone but discontinuous only on $\mathbb{Q}$, hence dense.

5. The convex conjugate need not be even defined, think of affine function $f(x) = ax - b$, its convex conjugate is only defined at $x^* = a$.

6. It can be any possible collection since we can take indicator function of desired set and declare it is continuous.

Section 5.1

15 Suppose no such two sequences exist, i.e. for any sequence (x_n) converge to 0, $(f(x_n))$ converge to some fixed constant ℓ independent of the sequence chosen. Then by sequential criterion, f is continuous at 0, i.e. limit at 0 exists, hence contradiction.

Section 5.2

6 For any $\epsilon > 0$, there exists $\delta' > 0$ such that for any y with $|y-b| < \delta'$, then $|g(y)-g(b)| < \epsilon$. Now take $\epsilon' = \delta'$, there exists $\delta > 0$ such that for any x with $|x - c| < \delta$, then $|f(x) - f(c)| < \delta$, hence $|g(f(x)) - g(f(b))| < \epsilon$ by combining two. Hence done for this $\epsilon - \delta$ on $g \circ f$.

11 by continuity and fact that $f(x_n) \geq g(x_n)$ for any n. Hence taking limit, by continuity, $f(c) \geq g(c)$, hence $c \in S$.

Section 5.3

16 g take whole real line to $(0, 1]$ half-closed and half open. It take compact interval to compact interval. It take open interval into open interval if the interval in domain not contain 0, else it maps to a left-open-right-closed interval. h take whole real line to whole real line. It preserves nature (open/closed/half/compact/unbounded) of interval.

18 Note I is closed and bounded interval, hence compact by Heine-Borel theorem. Hence for each point x on it, take neighbourhood I_x on which f is locally bounded, say by B_x. Then by compactness, this open cover has a finite subcover, hence we take maximum of corresponding bounds, hence forming a global bound.

Section 5.4

16 Provided f is Lipschitz, for any $\epsilon > 0$, take $\delta = \epsilon/2K$ where K is LIpschitz constant for f. Then for any pairwise disjoint interval $[x_k, y_k]$ with $\sum |x_k - y_k| < \delta$, then $\sum |f(x_k) - f(y_k)| \leq K \sum |x_k - y_k| < \epsilon$.

Section 5.6

7 This set can be seenas

$$\inf\{f(y) : y > c\} - \inf\{-f(x) : x < c\}$$

hence equals

$$\inf\{f(y) : y > c\} - \sup\{f(x) : x < c\}$$

hence same

Tutorial 12

Section 5.1

4 (a) Continuous except at all intergers.

(b) Continuous except at all nonzero intergers

(c) Continuous except at all $k\pi, 2k\pi + \frac{\pi}{2}$ for interger k.

(d) Continuous at all point except at inverse of nonzero integers.

14 Note that any open interval densely contains subset of rational number and is countably infinite. As with infinite number of rational numbers, their denominator must $\to \infty$. Hence on bounded domain, this should always contain numerator $\to \infty$. Hence k is unbounded on such numbers.

Section 5.2

1 (a) Continuous everywhere

(b) continuous everywhere

(c) continuous except at 0

(d) continuous everywhere

11 done above

15 The formula is easy to show, and each term in the formula is just continuous, hence h is continuous.

Section 5.3

3 Take any point x_1 in the interval. By the assumption, we can take x_2 such that $|f(x_2) \leq \frac{1}{2}|f(x_1)|$. And inductively we take (x_n) such that they fulfil such inequality, hence by continuity and that $|f(x_{n+1})| \leq f(x_1)/2^n$ hence take by Bolzano-Weiestrass theorem a convergent subsequence of (x_n), hence $(f(x_{n_k}))$ is convergent which by inequality converge to 0. Hence this limit point can be taken as c.

13 This is identical to solution to midterm 2 question 3.

Section 5.4

1 Since f defined on compact interval, f attains maximum and minimum. Moreover, by its monotone nature, the endpoint is the maximum/minimum obviously. Now if it is strictly increasing, then each value is attained only at most once, hence the endpoint is unique maximum/minimum.

12 Use bisection method for it, we show that f is strictly increasing on dense set of diadyic numbers. Inductively if it happens WLOG that $f(0) < f(1) \leq f(\frac{1}{2})$, by intermediate value theorem, there exists $c \in [0, \frac{1}{2}]$ taking value $f(1)$, hence contradiction to assumption. Now repeat this argument. Since f is strictly increasing on this dense set, hence increasing on whole interval. If it is not stirctly increasing somewhere on the interval, say $f(x) = f(y)$ then f is constant on interval $[x, y]$ hence contradiction.

16 Done above

Section 5.6

1 Note for any $\epsilon > 0$, take $\delta = \frac{\epsilon}{2a^2}$, then in any δ-interval, $|f(x) - f(y)| \leq \frac{1}{a^2}|x - y| < \epsilon$. Hence uniformly continuous.

12 Take $\epsilon - \delta'$ for continuity at 0, then take $\epsilon - \delta''$ for uniform continuity at $[\delta/2, \infty)$. Then take δ as minimum of them, this is the desired uniform continuity.

16 done above.

Final Comments

We trey to choose as much questions from each section of textbook exercise as possible, but this just meant to increase your awareness of importance of textbook exercise, you should try to do all textbook questions on your own. Note there are some section exercise not covered, for example chapter 1, section 2.1, 2.2, 3.6. Some section exercise are less covered, for example section 2.5, section 3.5. Besides, chapter 5 exercise are less picked that they only appear in auxiliary exericse, you better do more there to prepare for the midterm.