# Answer to tutorial 1-4 exercises

Selected.

## Tutorial 1

Q1: show $(a+b)^2 = a^2+2ab+b^2$ & $(-a)^2=a^2$.

Ans:
$$\begin{aligned}
(a+b)^2 &= (a+b)\cdot(a+b) && \text{by def.}\\
&=(a+b)\cdot a+(a+b)\cdot b && \text{by distributivity.}\\
&=a\cdot a+b\cdot a+a\cdot b+b\cdot b && \text{as above.}\\
&=a^2+b\cdot a+a\cdot b+b^2 && \text{by def.}\\
&=a^2+a\cdot b+a\cdot b+b^2 && \text{by commutativity.}\\
&=a^2+1\cdot(a\cdot b)+1\cdot(a\cdot b)+b^2 && \text{1 is multiplicative id.}\\
&=a^2+(1+1)\cdot(a\cdot b)+b^2 && \text{by reverse of distributivity.}\\
&=a^2+2\cdot(a\cdot b)+b^2. && \text{by def, } 1+1=:2.
\end{aligned}$$

$$\begin{aligned}
(-a)^2 &= (-a)\cdot(-a) && \text{by def}\\
&=((-1)\cdot a)\cdot((-1)\cdot a) && \text{by lemma below.}\\
&=(((-1)\cdot a)\cdot(-1))\cdot a && \text{by associativity.}\\
&=((-1)\cdot((-1)\cdot a))\cdot a && \text{by commutativity.}\\
&=(((-1)\cdot(-1))\cdot a)\cdot a && \text{by associativity.}\\
&=(1\cdot a)\cdot a && \text{by lemma.}\\
&=a\cdot a=a^2 && \text{by multiplicative id \& def.}
\end{aligned}$$

lemma: $-a=(-1)\cdot a$ & $(-1)\cdot(-1)=1$.

subpf: by uniqueness of additive inverse, (i.e. $a+b=0$ and $a+c=0$ $\Rightarrow b=c$ ).
one just need to show

$$(-1)\cdot a+a=0.$$

$$(-1)\cdot a+a=(-1)\cdot a+1\cdot a=((-1)+1)\cdot a=0\cdot a.$$

$$0\cdot a+0\cdot a=(0+0)\cdot a=0\cdot a$$

$$0\cdot a=0\cdot a+0\cdot a+(-(0\cdot a))=0\cdot a+(-(0\cdot a))=0.$$

hence $(-1)\cdot a=-a$. hence $(-1)\cdot(-1)=--1=1$

Tutorial 1

Q2: show that $1<2<3<\ldots$ and $(n,n+1)\cap\mathbb{N}=\emptyset \ \forall n\in\mathbb{N}$.

Ans: one need to formulate it rigorously to get a proof.

which is $P(i)$ proposition: $i<i+1<i+2<\ldots$

i.e. $Q(j)$ is true where $j$ starts from $i$

and

Proposition: if $j\geq i$, then $j+1>i$.

We verify: $P(1)$:

We verify $Q(1)$: we have $1\geq 1$, and $1>0$.

hence adding to $2:=1+1>1+0=1$

hence $Q(1)$ is true.

Let $j\in\mathbb{N}$. Suppose $j\geq 1$.

Suppose $Q(j)$ is true,

hence $j+1>1$ by $Q(j)$.

hence this satisfies requirement. $j+1\geq i$ of $Q(j+1)$,

hence also $1>0$.

adding to $j+2:=(j+1)+1>i+0=i$.

hence $Q(j+1)$ is true.

by principle of math induction, $Q(j)$ is true on ~~that domain~~ $\mathbb{N}$.

hence $P(1)$ is true.

Let $i\in\mathbb{N}$, Suppose $P(i)$ is true,

$i>0$ we verify $P(i+1)$:

we verify $Q(0)$: obviously $0<i$, hence $Q(0)$ is true.

Let $j\in\mathbb{N}$, suppose $j\geq i+1$, suppose $Q(j)$ is true,

then we verify $Q(j+1)$:

since $j\geq i+1$,

we have $j+1>j\geq i+1$,

now $1>0$

$j+2:=(j+1)+1>i+0+0=i+1$

hence $Q(j+1)$ is true.

by principle of math induction, $Q(j)$ is true on $\mathbb{N}$.

hence $P(i+1)$ is true

by principle of math induction, $P(i)$ is true on $\mathbb{N}$.

hence proven.

For second part,

Tutorial 1 Q2: (cont'd)

Ans: for second part, you are to show that there is no. integer between $n$ and $n+1$.

or you show that if $j > n \implies j \geq n+1$.

prove by induction:

$P(0)$: $j > 0$

then by previous prop, you know that $j$ should be in the latter list of number $1 < 2 < 3 < \dots$, which is meaning $j$ is inside the range of objects which one can attain by adding 1 to the integer 1, i.e. an inductive object.

as the previous prop show, $j \geq 0+1 = 1$ obviously.

now if $P(i)$ is true, we show $P(i+1)$:

if $j > i+1$, now $i \in \mathbb{N}$

hence $i > 0$

hence $j > 0+1 > 0$

hence $j$ should be obtained as an inductive object i.e. $j = j'+1$ for some $j' \in \mathbb{N}$.

$j'+1 > i+1$.

now we know $j' > i$

(if not then $j' \leq i$ hence $j'+1 \leq i+1$, contradiction)

hence by $P(i)$: $j' \geq i+1$

hence $j = j'+1 \geq i+1+1 = i+2$.

hence proven by MI.

now if $j \in (n, n+1) \cap \mathbb{N}$, $n \in \mathbb{N}$

then $j > n$, $j < n+1$.

but by prop $\Rightarrow$ $j \geq n+1$, contradiction.

hence $\nexists$ such $j$.

$\blacksquare$

# Tutorial 1

Q4: $X \neq \phi$, $\sup X = -\inf(-X)$

Ans: ~~$\forall a \in X$, $a \leq \sup X$~~

~~hence $-a \geq -\sup(X)$~~

~~but $-a \in -X$, hence~~

$\forall b \in (-X)$, $\exists a \in X$ s.t. $b = -a$.

hence $b = -a$.

$-b = a \in X$.

hence $-b \leq \sup X$. since sup is an upper bound

$b \geq -\sup X$. $\forall b \in (-X)$.

hence $-\sup X$ is a lower bound of set $(-X)$.

Since $\inf(-X)$ is defined as ~~LEAST~~ GREATEST lower bound of set $(-X)$,

hence as lower bound,

$\underset{\text{greatest lower bound}}{\inf(-X)} \geq \underset{\text{one of lower bound}}{-\sup(+X)}$.

$\forall a \in X$, since $X = [-(-X)]$

$\exists b \in (-X)$ s.t. $a = -b$.

hence $a = -b$

$-a = b \in (-X)$.

hence $-a \geq \inf(-X)$ since inf is a lower bound

~~Since~~ ~~$a \geq$~~

$a \leq -\inf(-X)$. $\forall a \in X$.

hence $-\inf(-X)$ is an upper bound of $(X)$,

Since $\sup(X)$ is defined as LEAST upper bound,

hence as upper bound,

$\underset{\text{least upper bound}}{\sup X} \leq \underset{\text{one of upper bound}}{-\inf(-X)}$

hence $-\sup X \geq \inf(-X)$.

hence $-\sup X = \inf(-X)$

$\sup X = -\inf(-X)$ $\blacksquare$

Tutorial 1

Q5: replace definition of limit of sequence with $\varepsilon$ by $5\varepsilon^2$, $N$ by $\sqrt{N}$, show they are equivalent.

Ans: the two statements are: (fix $l$)

$(*)$: $\forall \varepsilon>0$, $\exists N\in\mathbb{N}$ s.t. $\forall n\in\mathbb{N}$ s.t. $n\geq N$, we have $|x_n-l|<\varepsilon$.

$(**)$: $\forall \varepsilon>0$, $\exists N\in\mathbb{N}$ s.t. $\forall n\in\mathbb{N}$ s.t. $n\geq\sqrt{N}$, we have $|x_n-l|<5\varepsilon^2$.

we show $(*)\Rightarrow(**)$:

the reverse side is obviously stronger.

Suppose $(x_n)$ converge to $l$ in definition of $(*)$.

$\boxed{\forall\varepsilon>0,}$ (this $\varepsilon$ is for $\varepsilon$ in definition $(**)$.)

For $\varepsilon'=5\varepsilon^2$,

by definition of $(*)$,

$\exists N'\in\mathbb{N}$ s.t. $\forall m\in\mathbb{N}$ s.t. $m\geq N'$, we have $|x_m-l|<\varepsilon'$

$\boxed{\text{take } N=(N')^2}$.

now $\forall n\in\mathbb{N}$ s.t. $n\geq\sqrt{N}$.

then $n\geq\sqrt{N}=N'$,

hence by previous deduction, $\underline{|x_n-l|<\varepsilon'=5\varepsilon^2}$

hence we have $(x_n)$ converge to $l$ in definition $(**)$. ∎

Tutorial 2

EX 2.3.2: $S_2 = \{x \in \mathbb{R} : x > 0\}$. lower bound? upper bound? inf? sup?

Ans: ①: $\forall x \in S_2$,
by def, $x > 0$.
hence $0$ is a lower bound to $S_2$.
so $S_2$ has a lower bound.

②: suppose $S_2$ has an upper bound $M$,
then $\forall x \in S_2$, $M \geq x$.
hence $M \geq x > 0$.
hence $M$ is positive.
take $M' = M+1 > 0$.
hence $M' \in S_2$.
$M \geq M' \in S_2$ by definition
but $M' = M+1 > M$ obviously.
Contradiction. hence $S_2$ has no upper bound.

③. we show $0$ is the infimum of $S_2$.
by completeness, let $u = \inf S_2$. (since $S_2$ bounded below by $0$).
hence $\underset{\text{greatest lower bound}}{u} \geq \underset{\text{one of lower bound}}{0}$

now suppose $u > 0$ (i.e. $u \neq 0$).
then consider $u' = \frac{u}{2} > 0$ hence $u' \in S_2$.
by def of inf (as lower bound),
$u \leq u' \in S_2$.
but $u' = \frac{u}{2} < u$
Contradiction.
hence $u = 0$.

④. Since $S_2$ has no upper bound,
it cannot have supremum.
since supremum is an upper bound.

$\boxtimes$

Tutorial 2

Ex 2.3.14: $S$ bounded below.
lower bound $w$ of $S$ is $\inf S$ iff $\forall \varepsilon > 0$, $\exists t \in S$ s.t. $t < w + \varepsilon$.

Ans: with $w$ lower bound of $S$ fixed,

if $w$ is the $\inf S$, then $w$ is the greatest among all lower bound,

i.e. if $u > w$,
then $u$ cannot be a lower-bound of $S$,

hence $\forall \varepsilon > 0$, $w + \varepsilon > w$
hence $w + \varepsilon$ cannot be a lower-bound of $S$, failing to bound $t \in S$.
hence $\exists t \in S$ s.t. $t < w + \varepsilon$.

Conversely, suppose for contradiction that $w$ is not the $\inf S$.
hence $w$ is not the greatest among all lower bound,
hence ~~by~~ by completeness,
let $w' = \inf S$,
$w'$ is greatest among all lower-bound of $S$,
hence $w' > w$.

~~now let $\varepsilon$~~
now let $\varepsilon = w' - w > 0$,
by assumption, $\exists t \in S$ s.t. $t < w + \varepsilon = w + (w' - w) = w'$.
hence $w'$ fail to bound some $t \in S$.
hence $w'$ fail to be even a lower bound of $S$,

hence contradiction.
hence $w$ is the inf of $S$. ☒

Ex. 2.4.12

prove that $\sup_{x,y} h(x,y) = \sup_x(\sup_y h(x,y)) = \sup_y(\sup_x(h(x,y)))$.

Ans. we only show one side, $\sup_{x,y} = \sup_x(\sup_y)$.

$\forall x \in X$, $\forall y \in Y$, $h(x,y) \leq \sup_y h(x,y)$ (fixing $x$ here) vary over $y$.

the RHS become a function of $x$ only,
denote it as $F(x)$,
hence $h(x,y) \leq F(x)$. $\forall x, y$.
vary over $x$.
We have $h(x,y) \leq \sup_x F(x)$.
hence $\sup_x F(x) = \sup_x(\sup_y h(x,y))$ is an upper bound of all $h(x,y)$,
hence $\sup_{x,y} h(x,y) \leq \sup_x(\sup_y(h(x,y))$.

Tutorial 2 (cont'd)

Ex 2.4.12

Ans: Conversely, consider the set of all values, denoted in short hand $h(X, Y)$.

now if we fix $a \in X$, $\forall y \in Y$

then $h(a, y) \in h(X, Y)$.

collect them into set, written $h(a, Y) \subseteq h(X, Y)$. are sets

hence (sup of subset $\leq$ sup of set).

$\sup(h(a, Y)) \leq \sup(h(X, Y))$ ← sup of set.

but the LHS is then a function of a, denoted $F(a)$,

hence $F(a) \leq \sup h(X, Y)$. $\forall a \in X$.

hence $\sup h(X, Y)$ is an upper bound to all such $F(a)$,

hence $\sup h(X, Y) \geq \sup_a F(a)$

$= \sup_a (\sup_b (h(a, b)))$.

hence conclude the equation. $\blacksquare$

Ex. 2.5.8 : show $\bigcap J_n = \phi$ $(J_n = (0, \frac{1}{n}), n \in \mathbb{N}_x)$.

Ans: suppose $a \in \bigcap J_n$,

then $a \in J_n$ $\forall n \in \mathbb{N}_x$

hence $0 < a < \frac{1}{n}$ $\forall n \in \mathbb{N}_x$

hence ~~suppose~~ $a > 0$, by Archimedean principle, $\exists m \in \mathbb{N}$ s.t. $\frac{1}{m} < a$. , $m > 0$.

hence $a > \frac{1}{m}$, hence $a \notin J_m$.

Contradiction.

hence $\bigcap J_n = \phi$. $\blacksquare$

Tutorial 2.

Ex 3.1.12. show $\lim(\sqrt{n^2+1}-n)=0$.

Ans: before doing pf, do rough work here.

want $|(\sqrt{n^2+1}-n)-0|<\varepsilon$.

$\sqrt{n^2+1}-n<\varepsilon$.

$(n^2+1)-2n\sqrt{n^2+1}+n^2<\varepsilon^2$.

$2n^2+1-2n\sqrt{n^2+1}<\varepsilon^2$.

$(2n^2+1)-\varepsilon^2<2n\sqrt{n^2+1}$.

$(2n^2+1)^2-2\varepsilon^2(2n^2+1)+\varepsilon^4<4n^2(n^2+1)$.

$4n^4+4n^2+1-2\varepsilon^2(2n^2+1)+\varepsilon^4<4n^4+4n^2$.

$1-2\varepsilon^2(2n^2+1)+\varepsilon^4<0$.

$\varepsilon^4+1<2\varepsilon^2(2n^2+1)$.

Roughwork area

since n expected to be large, assume RHS > 4

take $\varepsilon$ small ⇝ obviously this is true.

so need $2\varepsilon^2(2n^2)>4$.

$n>\frac{1}{\varepsilon}$ is a good choice

also mind the case

if $\varepsilon$ is not small. (say $\varepsilon\geq 1$).

then we need $2\varepsilon^2(2n^2)>\varepsilon^4$

$n>\varepsilon$.

proceed with it

$\forall\varepsilon>0$, take. $N=\max\{\lceil\frac{1}{\varepsilon}\rceil,\lceil\varepsilon\rceil\}+1$

(no need take ceiling function, just use archimedean prnple to take integer $\gg\frac{1}{\varepsilon},\varepsilon$.)

$\forall n\geq N$, divide into 2 cases:

case 1: $\varepsilon\geq 1$, then $\varepsilon^4+1\leq\varepsilon^2\cdot n^2+\varepsilon^2$.

$\leq 2\varepsilon^2\cdot 2n^2+2\varepsilon^2=2\varepsilon^2(2n^2+1)$

proceed backward of above rough work.

$\varepsilon^4+1<2\varepsilon^2(2n^2+1)$.

$1-2\varepsilon^2(2n^2+1)+\varepsilon^4<0$.

$4n^4+4n^2+1-2\varepsilon^2(2n^2+1)+\varepsilon^4<4n^4+4n^2$.

$(2n^2+1)^2-2\varepsilon^2(2n^2+1)+\varepsilon^4<4n^2(n^2+1)$.

$((2n^2+1)-\varepsilon^2)^2<(2n\sqrt{n^2+1})^2$. since $2n^2+1>\varepsilon^2$.

$2n^2+1-\varepsilon^2<2n\sqrt{n^2+1}$.

$(n^2+1)-2n\sqrt{n^2+1}+n^2<\varepsilon^2$.

$(\sqrt{n^2+1}-n)^2<\varepsilon^2$ ⇝ $\sqrt{n^2+1}-n<\varepsilon$ since $n^2+1>n^2$, $\sqrt{n^2+1}>n$.

$|(\sqrt{n^2+1}-n)-0|<\varepsilon$.

done

(Cont'd)

Tutorial 2

Ex 3.1.12 case 2: $\varepsilon < 1, \quad n > \frac{1}{\varepsilon}$.

Ans:

$$\text{hence} \quad 2\varepsilon^2(2n^2+1) > 4 + 2\varepsilon^2 > 1 + 2\varepsilon^4 > 1 + \varepsilon^4.$$

$$\text{hence} \quad \varepsilon^4 + 1 < 2\varepsilon^2(2n^2+1).$$

$$1 - 2\varepsilon^2(2n^2+1) + \varepsilon^4 < 0.$$

$$(4n^4 + 4n^2 + 1) - 2\varepsilon^2(2n^2+1) + \varepsilon^4 < 4n^4 + 4n^2.$$

$$(2n^2+1)^2 - 2\varepsilon^2(2n^2+1) + \varepsilon^4 < 4n^2(n^2+1)$$

$$\left((2n^2+1) - \varepsilon^2\right)^2 < \left(2n\sqrt{n^2+1}\right)^2.$$

$$2n^2 + 1 - \varepsilon^2 < 2n\sqrt{n^2+1} \qquad \text{since} \quad 1 > \varepsilon^2.$$

$$(n^2+1) - 2n\sqrt{n^2+1} + n^2 < \varepsilon^2.$$

$$\left(\sqrt{n^2+1} - n\right)^2 < \varepsilon^2$$

$$\sqrt{n^2+1} - n < \varepsilon. \qquad \text{since} \quad n^2+1 > n^2. \quad \sqrt{n^2+1} > n.$$

$$\left|\left(\sqrt{n^2+1} - n\right) - 0\right| < \varepsilon.$$

hence the in both case. $\blacksquare$

~~Ex 3.1.16: show~~ $\cancel{\lim \frac{n^2}{n!} = 0}$.

~~Ans: Do roughwork first:~~

~~Roughwork~~

~~wait~~ $\cancel{\left|\frac{n^2}{n!} - 0\right| < \varepsilon}$.

$\cancel{\frac{n^2}{n!} < \varepsilon}$.

$\cancel{\frac{n}{(n-1)!} < \varepsilon}$ ~~wait~~ $\cancel{n > 1}$.

~~hence~~ $\cancel{\frac{1}{(n-2)!}(2) < \varepsilon}$ ~~since~~ $\cancel{2(n-1) > n}$

~~show stronger.~~ $\cancel{2 > \frac{n}{n-1}}$

Tutorial 3.

**HW2.4.** show $\sup(A+B) = \sup A + \sup B$ if either side exists.

Ans: ~~$\forall$~~ let $a := \sup A$, $b := \sup B$, $e := \sup(A+B)$.

$\forall x \in (A+B)$, $\exists\, c \in A, d \in B$ s.t. $x = c+d$.

hence $x = c+d$ ~~$\leq a+b$~~

$\leq a+d$ since $a \geq c\ \forall c \in A$.

$\leq a+b$ since $b \geq d\ \forall d \in B$.

hence $a+b$ constant is an upper bound to any $x \in (A+B)$, hence to whole set $(A+B)$,

hence $\underbrace{a+b}_{\text{one of upper bound}} \geq \underbrace{\sup(A+B)}_{\text{least upper bound}} =: e$

Conversely,

$\forall c \in A, d \in B$,

note that $(A+B) = \bigcup_{c \in A} (c+B)$. cosets.

hence fix $c \in A$, $c+B \subseteq A+B$.

hence $\sup(c+B) \leq \sup(A+B) =: e$.

$c + \sup B \leq e$.

$c + b \leq e \quad \forall c \in A$.

hence $c \leq e-b$

$e-b$ constant is hence an upper bound to any $c \in A$

hence.

$\underbrace{e-b}_{\text{one of upper bound}} \geq \underbrace{\sup A}_{\text{least upper bound}} = a$.

hence $e \geq a+b$.

hence done. $\square$

**Ex 3.2.5** show they are not convergent.

$(2^n)$, $((-1)^n n^2)$.

Ans: (a): $\forall \ell \in \mathbb{R}$, (we show $\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists n \geq N$ s.t. $|x_n - \ell| \geq \varepsilon$)

take $\varepsilon = 1$, $\forall N \in \mathbb{N}$,

take by Archimedean principle. $n$ s.t. $n \geq N$,

$n > |\ell| + 2$.

(Cont'd)

tutorial 3.

Ex 3.2.5

Ans: (a): with $n \geq N_1$, $n > |l| + 1$.

hence $2^n > 2^{|l|+1} = 2 \cdot 2^{|l|} = 2^{|l|} + 2^{|l|}$
$\geq 2^{|l|} + 1 \geq |l| + 1 \geq l + 1$.

hence $|2^n - l| \geq 1$.

hence is divergent.

(b): $\forall l \in \mathbb{R}$,
take $\varepsilon = 1$, $\forall N \in \mathbb{N}$,
take $n \geq N$ and $n \geq |l| + 2$ and $n$ is even.

hence ~~$(-1)^n n^2 \geq (-1)^n$~~
$(-1)^n n^2 = n^2$
$\geq (|l|+2)^2 = |l|^2 + 4|l| + 4 > |l| + 1$.
$\geq l + 1$

hence ~~$(-1)^n n^2 - (|l|+1) >$~~
$|(-1)^n n^2 - (l)| \geq 1$.

hence divergent.

Ex 3.2.11 determine limit of $(3\sqrt{n})^{\frac{1}{2n}}$, $(n+1)^{\frac{1}{\ln(n+1)}}$

Ans: (a): do roughwork first.

want $|(3\sqrt{n})^{\frac{1}{\sqrt{2n}}} - 1| < \varepsilon$.

$(3\sqrt{n})^{\frac{1}{\sqrt{2n}}} < \varepsilon + 1$.

$3\sqrt{n} < (1+\varepsilon)^{\sqrt{2n}}$.

show stronger statement by binomial series

$3\sqrt{n} < 1 + \sqrt{2n}\,\varepsilon + \frac{\sqrt{2n}(\sqrt{2n}-1)}{2}\varepsilon^2$

take $n$ s.t. $\frac{\sqrt{2n}-1}{2}\varepsilon^2 > 3$.

if $\varepsilon \geq 1$. then need $\frac{\sqrt{2n}-1}{2} > 3$.

$n > 49$.

$\forall \varepsilon > 0$, take $N$ s.t. $N > \left(\frac{6}{\varepsilon^2} + 1\right)^2 / 2$, and $N > 49$, hence $\frac{\sqrt{2N}-1}{2}\varepsilon^2 > 3$.

$\forall n \geq N$.

(Cont'd)

Tutorial 3

Ex 3.2.11

Ans (a):

Case 1: $\varepsilon \geq 1$,

hence with $N > 49$, $n \geq N$.

hence $\frac{\sqrt{2n}-1}{2} > 3$

hence $3\sqrt{n} < \frac{\sqrt{2n}(\sqrt{2n}-1)}{2}\varepsilon^2 < (1+\sqrt{2n}\,\varepsilon) + \frac{\sqrt{2n}(\sqrt{2n}-1)}{2}\varepsilon^2 < (1+\varepsilon)^{\sqrt{2n}}$.

$3\sqrt{n} < (1+\varepsilon)^{\sqrt{2n}}$.

$(3\sqrt{n})^{\frac{1}{\sqrt{2n}}} < \varepsilon + 1$.

$|(3\sqrt{n})^{\frac{1}{\sqrt{2n}}} - 1| < \varepsilon$. done.

Case 2: $\varepsilon < 1$,

$n \geq N > (\frac{6}{\varepsilon^2}+1)^2/2$ hence $\frac{\sqrt{2n}-1}{2}\varepsilon^2 > 3$.

hence. $3\sqrt{n} < \sqrt{n}\left(\frac{\sqrt{2n}-1}{2}\right)\varepsilon^2 < 1+\sqrt{2n}\,\varepsilon + \frac{\cancel{\sqrt{n}}(\sqrt{n}-1)}{2}\varepsilon^2$

$3\sqrt{n} < (1+\varepsilon)^{\sqrt{2n}}$ $< (1+\varepsilon)^{\sqrt{2n}}$.

proceed as above.

(b):

$$(n+1)^{\frac{1}{\log(n+1)}} = e^{\frac{\log(n+1)}{\log(n+1)}} = 1$$

$\square$

hence $\forall \varepsilon > 0$, take $N=1$,

then $\forall n \geq N$,

$|(n+1)^{\frac{1}{\log(n+1)}} - 1| = |1-1| = 0 < \varepsilon$.

hence done. $\square$

Tutorial 4

Ex 3.3.2 $x_1 > 1$, $x_{n+1} = 2 - \frac{1}{x_n}$ $\forall n \in \mathbb{N}$
show it is bounded & monotone & limit.

Ans.

Suppose it is bounded & monotone,
by monotone convergence thm, $\exists$ limit $l$,
$\rightsquigarrow$ taking limit on both side

$$x_{n+1} = 2 - \frac{1}{x_n}$$
$$l = 2 - \frac{1}{l}$$
$$l^2 - 2l + 1 = 0$$
$$l = 1.$$

( Justification:
$X^+ := (x_{n+1})$ as 1-tail of sequence $X = (x_n)$.
$X^+$ & $X$ has same limit.
and limit of $X^+ = (x_{n+1}) = (2 - \frac{1}{x_n})$
$$= \lim(2) - \lim \frac{1}{x_n}$$
$$= 2 - \frac{1}{\lim x_n}$$
by limit thm.

hence we have $\lim X^+ = 2 - \frac{1}{\lim X}$ (assume $\lim X > 0$.)
$\overset{\shortparallel}{\lim X}$
$l = 2 - \frac{1}{l}$ ).

Show bounded:
inductively $x_i \in (1, 2)$.
$P(1)$: $x_1 > 1$.
$P(i)$ true
$\rightsquigarrow P(i+1)$: $x_{i+1} = 2 - \frac{1}{x_i}$
$> 2 - 1 = 1$
and $x_{i+1} = 2 - \frac{1}{x_i} < 2$ obviously.

Show monotone: inductively
$P(1)$: $x_2 - x_1 = 2 - \frac{1}{x_1} - x_1 = \frac{2x_1 - 1 - x_1^2}{x_1}$
$$= \frac{-(x_1-1)^2}{x_1} < 0.$$
hence $x_2 < x_1$

$P(i)$ true $\Rightarrow$
$P(i+1)$: $x_{i+1} - x_i$
$$= 2 - \frac{1}{x_i} - x_i$$
$$= -\frac{1}{x_i}(x_i^2 - 2x_i + 1) = -\frac{(x_i - 1)^2}{x_i} < 0.$$
hence $x_{n+1} < x_n$.

hence decreasing.

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