MATH2050B tutorial 07

October 23, 2022

Abstract

This is for limit of functions and provides counterexamples.

Counterexamples

This section is to provide or let you try counterexamples for various analysis concepts.

1. no Completeness/ Monotone convergence theorem/ Cauchy criterion

   the space of rational numbers is not complete, think of any rational sequence converging to $\sqrt{2}$

2. no sequential criterion

   the space of function with uniform convergence, take a pointwise convergent sequence of functions but not uniformly convergent

3. convergent but not contractive sequence
4. a sequence with limits (from subsequences) from no other limsup and liminf
5. a sequence with 3 limits (from subsequence)
6. a sequence with any number of limits
7. (not possible) a sequence with no monotone subsequence

Section 4.1

9. Use either the ε-δ definition of limit or the Sequential Criterion for limits, to establish the following limits.

   (a) $\displaystyle\lim_{x \to 2} \frac{1}{1-x} = -1$, (b) $\displaystyle\lim_{x \to 1} \frac{x}{1+x} = \frac{1}{2}$,

   (c) $\displaystyle\lim_{x \to 0} \frac{x^2}{|x|} = 0$, (d) $\displaystyle\lim_{x \to 1} \frac{x^2 - x + 1}{x+1} = \frac{1}{2}$.

10. Use the definition of limit to show that

    (a) $\displaystyle\lim_{x \to 2} (x^2 + 4x) = 12$, (b) $\displaystyle\lim_{x \to -1} \frac{x+5}{2x+3} = 4$.

11. Use the definition of limit to prove the following.

    (a) $\displaystyle\lim_{x \to 3} \frac{2x+3}{4x-9} = 3$, (b) $\displaystyle\lim_{x \to 6} \frac{x^2 - 3x}{x+3} = 2$.

12. Show that the following limits do *not* exist.

    (a) $\displaystyle\lim_{x \to 0} \frac{1}{x^2} \quad (x > 0)$, (b) $\displaystyle\lim_{x \to 0} \frac{1}{\sqrt{x}} \quad (x > 0)$,

    (c) $\displaystyle\lim_{x \to 0} (x + \operatorname{sgn}(x))$, (d) $\displaystyle\lim_{x \to 0} \sin(1/x^2)$.

Section 4.2

1. Apply Theorem 4.2.4 to determine the following limits:

   (a) $\displaystyle\lim_{x \to 1} (x+1)(2x+3) \quad (x \in \mathbb{R})$, (b) $\displaystyle\lim_{x \to 1} \frac{x^2+2}{x^2-2} \quad (x > 0)$,

   (c) $\displaystyle\lim_{x \to 2} \left(\frac{1}{x+1} - \frac{1}{2x}\right) \quad (x > 0)$, (d) $\displaystyle\lim_{x \to 0} \frac{x+1}{x^2+2} \quad (x \in \mathbb{R})$.

2. Determine the following limits and state which theorems are used in each case. (You may wish to use Exercise 15 below.)

   (a) $\displaystyle\lim_{x \to 2} \sqrt{\frac{2x+1}{x+3}} \quad (x > 0)$, (b) $\displaystyle\lim_{x \to 2} \frac{x^2-4}{x-2} \quad (x > 0)$,

   (c) $\displaystyle\lim_{x \to 0} \frac{(x+1)^2 - 1}{x} \quad (x > 0)$, (d) $\displaystyle\lim_{x \to 1} \frac{\sqrt{x} - 1}{x-1} \quad (x > 0)$.

3. Find $\displaystyle\lim_{x \to 0} \frac{\sqrt{1+2x} - \sqrt{1+3x}}{x + 2x^2}$ where $x > 0$.

4. Prove that $\displaystyle\lim_{x \to 0} \cos(1/x)$ does not exist but that $\displaystyle\lim_{x \to 0} x\cos(1/x) = 0$.

Section 4.3

4. Let $c \in \mathbb{R}$ and let f be defined for $x \in (c, \infty)$ and $f(x) > 0$ for all $x \in (c, \infty)$. Show that $\displaystyle\lim_{x \to c} f = \infty$ if and only if $\displaystyle\lim_{x \to c} 1/f = 0$.

5. Evaluate the following limits, or show that they do not exist.

   (a) $\displaystyle\lim_{x \to 1+} \frac{x}{x-1} \quad (x \neq 1)$, (b) $\displaystyle\lim_{x \to 1} \frac{x}{x-1} \quad (x \neq 1)$,

   (c) $\displaystyle\lim_{x \to 0+} (x+2)/\sqrt{x} \quad (x > 0)$, (d) $\displaystyle\lim_{x \to \infty} (x+2)/\sqrt{x} \quad (x > 0)$,

   (e) $\displaystyle\lim_{x \to 0} (\sqrt{x+1})/x \quad (x > -1)$, (f) $\displaystyle\lim_{x \to \infty} (\sqrt{x+1})/x \quad (x > 0)$,

   (g) $\displaystyle\lim_{x \to \infty} \frac{\sqrt{x} - 5}{\sqrt{x} + 3} \quad (x > 0)$, (h) $\displaystyle\lim_{x \to \infty} \frac{\sqrt{x} - x}{\sqrt{x} + x} \quad (x > 0)$.