MATH2050B tutorial 05

October 16, 2022

Abstract

This is for limit superior, series, function limits. The questions to work on are:
new questions (for limit superior),
3.7: 10-18 (for series),
4.13: 9-12

New Questions

1. Find the limit, limit superior/inferior for each of the following sequences (a_n) (if they exist):
   (a) $a_n = \frac{(-1)^n}{n+1}$
   (b) $a_n = n(-1)^n$
   (c) $a_n = (-1)^n + (-1)^{n+1}$
   (d) $a_n = (-1)^n + (-1)^{n+2}$
   (e) $a_n = n\sin(\frac{2}{n})$
   (f) $a_n = \sin(m\pi) + \cos(m\pi)$
   (g) $a_n = 2(-1)^n + \frac{n}{n+1}$
   (h) $a_n = \sin n$
   (i) $a_n = \tan n$
   (j) $a_n = \frac{n}{\log n}$
   (k) $a_n = \prod_{i=1} n \sin n$
   (l) $a_n = n\sin(\frac{n}{2})$
2. If sequence (a_n) is convergent and (b_n) arbitrary sequence, then

$$\limsup(a_n + b_n) = \limsup a_n \ + \ \limsup b_n$$

3. Prove that if sequence (a_n) with each term nonzero such that $\liminf \frac{a_{n+1}}{a_n} = \ell > 1$, then (a_n) is divergent. Can it contain a convergent subsequence? What if $\ell = 1$?

Section 3.7

10. Use an argument similar to that in Example 3.7.6(f) to show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent.

11. If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum a_n^2$ always convergent? Either prove it or give a counterexample.

12. If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum \sqrt{a_n}$ always convergent? Either prove it or give a counterexample.

13. If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum \sqrt{a_n a_{n+1}}$ always convergent? Either prove it or give a counterexample.

14. If $\sum a_n$ with $a_n > 0$ is convergent, and if $b_n := (a_1 + \cdots + a_n)/n$ for $n \in \mathbb{N}$, then show that $\sum b_n$ is always divergent.

15. Let $\sum_{n=1}^{\infty} a(n)$ be such that $(a(n))$ is a decreasing sequence of strictly positive numbers. If $s(n)$ denotes the nth partial sum, show (by grouping the terms in $s(2^n)$ in two different ways) that

$\frac{1}{2}(a(1) + 2a(2) + \cdots + 2^n a(2^n)) \leq s(2^n) \leq \big(a(1) + 2a(2) + \cdots + 2^{n-1} a(2^{n-1})\big) + a(2^n).$

Use these inequalities to show that $\sum_{n=1}^{\infty} a(n)$ converges if and only if $\sum_{n=1}^{\infty} 2^n a(2^n)$ converges. This result is often called the **Cauchy Condensation Test**; it is very powerful.

16. Use the Cauchy Condensation Test to discuss the p-series $\sum_{n=1}^{\infty} (1/n^p)$ for $p > 0$.

17. Use the Cauchy Condensation Test to establish the divergence of the series:

(a) $\sum \frac{1}{n \ln n}$, (b) $\sum \frac{1}{n(\ln n)(\ln \ln n)}$,

(c) $\sum \frac{1}{n(\ln n)(\ln \ln n)(\ln \ln \ln n)}$.

18. Show that if $c > 1$, then the following series are convergent:

(a) $\sum \frac{1}{n(\ln n)^c}$, (b) $\sum \frac{1}{n(\ln n)(\ln \ln n)^c}$.

Section 4.1

9. Use either the ε-δ definition of limit or the Sequential Criterion for limits, to establish the following limits.

(a) $\lim\limits_{x\to 2} \dfrac{1}{1-x} = -1$,

(b) $\lim\limits_{x\to 1} \dfrac{x}{1+x} = \dfrac{1}{2}$,

(c) $\lim\limits_{x\to 0} \dfrac{x^2}{|x|} = 0$,

(d) $\lim\limits_{x\to 1} \dfrac{x^2-x+1}{x+1} = \dfrac{1}{2}$.

10. Use the definition of limit to show that

(a) $\lim\limits_{x\to 2} \left(x^2+4x\right) = 12$,

(b) $\lim\limits_{x\to -1} \dfrac{x+5}{2x+3} = 4$.

11. Use the definition of limit to prove the following.

(a) $\lim\limits_{x\to 3} \dfrac{2x+3}{4x-9} = 3$,

(b) $\lim\limits_{x\to 6} \dfrac{x^2-3x}{x+3} = 2$.

12. Show that the following limits do *not* exist.

(a) $\lim\limits_{x\to 0} \dfrac{1}{x^2} \quad (x>0)$,

(b) $\lim\limits_{x\to 0} \dfrac{1}{\sqrt{x}} \quad (x>0)$,

(c) $\lim\limits_{x\to 0} (x+\operatorname{sgn}(x))$,

(d) $\lim\limits_{x\to 0} \sin\left(1/x^2\right)$.