MATH2050B tutorial 04

October 5, 2022

Abstract

The questions to work on are:
tutorial page: Q1 (on Archimedean properties)
3.2: 4/5/6/22
3.3: 1/2/3
3.4: 12/13/17/18/19
3.5: 1/2/3/12/13

Tutorial Page

1. Prove that the following are equivalent formulation of the Archimedean properties (fromulate them rigorously):
   - One can find integer larger than a given real number.
   - There is no upper bound to set of all natural numbers.
   - One can find arbitrarily small reciprocal of natural numbers.
   - If $a < b$, one can always find N such that $a + \frac{1}{N} < b$.
   - If $0 < a < b$, one can find N such that $Na > b$.
   - If $1 < a < b$, one can find N such that $a^N > b$.

Section 3.2

4. Show that if X and Y are sequences such that X converges to $x \neq 0$ and XY converges, then Y converges.

5. Show that the following sequences are not convergent.

   (a) (2^n), (b) $((-1)^n n^2)$.

6. Find the limits of the following sequences:

   (a) $\lim\left((2 + 1/n)^2\right)$, (b) $\lim\left(\dfrac{(-1)^n}{n+2}\right)$,

   (c) $\lim\left(\dfrac{\sqrt{n}-1}{\sqrt{n}+1}\right)$, (d) $\lim\left(\dfrac{n+1}{n\sqrt{n}}\right)$.

22. Suppose that (x_n) is a convergent sequence and (y_n) is such that for any $\varepsilon > 0$ there exists M such that $|x_n - y_n| < \varepsilon$ for all $n \geq M$. Does it follow that (y_n) is convergent?

Section 3.3

1. Let $x_1 := 8$ and $x_{n+1} := \frac{1}{2}x_n + 2$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Find the limit.

2. Let $x_1 > 1$ and $x_{n+1} := 2 - 1/x_n$ for $n \in \mathbb{N}$. show that (x_n) is bounded and monotone. Find the limit.

3. Let $x_1 \geq 2$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.

Section 3.4

12. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$.

13. If $x_n := (-1)^n/n$, find the subsequence of (x_n) that is constructed in the second proof of the Bolzano-Weierstrass Theorem 3.4.8, when we take $I_1 := [-1, 1]$.

17. Alternate the terms of the sequences $(1 + 1/n)$ and $(-1/n)$ to obtain the sequence (x_n) given by

$$(2, -1, 3/2, -1/2, 4/3, -1/3, 5/4, -1/4, \ldots).$$

Determine the values of $\lim\sup(x_n)$ and $\lim\inf(x_n)$. Also find $\sup\{x_n\}$ and $\inf\{x_n\}$.

18. Show that if (x_n) is a bounded sequence, then (x_n) converges if and only if $\lim\sup(x_n) = \lim\inf(x_n)$.

19. Show that if (x_n) and (y_n) are bounded sequences, then

$$\lim\sup(x_n + y_n) \leq \lim\sup(x_n) + \lim\sup(y_n).$$

Give an example in which the two sides are not equal.

Section 3.5

1. Give an example of a bounded sequence that is not a Cauchy sequence.

2. Show directly from the definition that the following are Cauchy sequences.

   (a) $\left(\frac{n+1}{n}\right)$, (b) $\left(1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)$.

3. Show directly from the definition that the following are not Cauchy sequences.

   (a) $((-1)^n)$, (b) $\left(n + \frac{(-1)^n}{n}\right)$, (c) $(\ln n)$

12. If $x_1 > 0$ and $x_{n+1} := (2 + x_n)^{-1}$ for $n \geq 1$, show that (x_n) is a contractive sequence. Find the limit.

13. If $x_1 := 2$ and $x_{n+1} := 2 + 1/x_n$ for $n \geq 1$, show that (x_n) is a contractive sequence. What is its limit?