MATH2050B Tutorial 2

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Abstract

This tutorial aims to provide exercise for familiarity with different new concepts in analysis. Exercises are
2.3.1/2/14,
2.4.1/2/11/12/16,
2.5.7/8/9,
3.1.12/16/17/18,
3.2.2/6/19.

Section 2.3

Exercise 1, 2, 14.

1. Let $S_1 := \{x \in \mathbb{R} : x \geq 0\}$. Show in detail that the set S_1 has lower bounds, but no upper bounds. Show that $\inf S_1 = 0$.
2. Let $S_2 := \{x \in \mathbb{R} : x > 0\}$. Does S_2 have lower bounds? Does S_2 have upper bounds? Does $\inf S_2$ exist? Does $\sup S_2$ exist? Prove your statements.

14. Let S be a set that is bounded below. Prove that a lower bound w of S is the infimum of S if and only if for any $\varepsilon > 0$ there exists $t \in S$ such that $t < w + \varepsilon$.

Section 2.4

Exercise 1, 2, 11, 12, 16.

1. Show that $\sup\{1 - 1/n : n \in \mathbb{N}\} = 1$.
2. If $S := \{1/n - 1/m : n, m \in \mathbb{N}\}$, find $\inf S$ and $\sup S$.

11. Let X and Y be nonempty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be defined by

$$f(x) := \sup\{h(x,y) : y \in Y\}, \quad g(y) := \inf\{h(x,y) : x \in X\}.$$

Prove that

$$\sup\{g(y) : y \in Y\} \leq \inf\{f(x) : x \in X\}.$$

We sometimes express this by writing

$$\sup_y \inf_x h(x,y) \leq \inf_x \sup_y h(x,y).$$

Note that Exercises 9 and 10 show that the inequality may be either an equality or a strict inequality.

12. Let X and Y be nonempty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $F : X \to \mathbb{R}$ and $G : Y \to \mathbb{R}$ be defined by

$$F(x) := \sup\{h(x,y) : y \in Y\}, \quad G(y) := \sup\{h(x,y) : x \in X\}.$$

Establish the **Principle of the Iterated Suprema:**

$$\sup\{h(x,y) : x \in X, y \in Y\} = \sup\{F(x) : x \in X\} = \sup\{G(y) : y \in Y\}$$

We sometimes express this in symbols by

$$\sup_{x,y} h(x,y) = \sup_x \sup_y h(x,y) = \sup_y \sup_x h(x,y).$$

16. Modify the argument in Theorem 2.4.7 to show that if $a > 0$, then there exists a positive real number z such that $z^2 = a$.

Section 2.5

Exercise 7, 8, 9.

7. Let $I_n := [0, 1/n]$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} I_n = \{0\}$.
8. Let $J_n := (0, 1/n)$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} J_n = \emptyset$.
9. Let $K_n := (n, \infty)$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} K_n = \emptyset$.

Section 3.1

Exercise 12, 16, 17, 18.

12. Show that $\lim(\sqrt{n^2+1} - n) = 0$.

16. Show that $\lim(n^2/n!) = 0$.

17. Show that $\lim(2^n/n!) = 0$. [*Hint*: If $n \geq 3$, then $0 < 2^n/n! \leq 2\left(\frac{2}{3}\right)^{n-2}$.]

18. If $\lim(x_n) = x > 0$, show that there exists a natural number K such that if $n \geq K$, then $\frac{1}{2}x < x_n < 2x$.

Section 3.2

Exercise 2, 6, 19.

2. Give an example of two divergent sequences X and Y such that:
   (a) their sum $X + Y$ converges, (b) their product XY converges.

3. Show that if X and Y are sequences such that X and $X + Y$ are convergent, then Y is convergent.

6. Find the limits of the following sequences:

   (a) $\lim\left((2 + 1/n)^2\right)$, (b) $\lim\left(\dfrac{(-1)^n}{n+2}\right)$,

   (c) $\lim\left(\dfrac{\sqrt{n}-1}{\sqrt{n}+1}\right)$, (d) $\lim\left(\dfrac{n+1}{n\sqrt{n}}\right)$.

19. Discuss the convergence of the following sequences, where a, b satisfy $0 < a < 1, b > 1$.
    (a) $(n^2 a^n)$, (b) (b^n/n^2),
    (c) $(b^n/n!)$, (d) $(n!/n^n)$.