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We note that if we reverse the order, then the composition $f \circ g$ is given by the formula

$$(f \circ g)(x) = 1 - x,$$

but only for those x in the domain $D(g) = \{x : x \geq 0\}$. □

We now give the relationship between composite functions and inverse images. The proof is left as an instructive exercise.

1.1.14 Theorem *Let $f: A \to B$ and $g: B \to C$ be functions and let H be a subset of C. Then we have*

$$(g \circ f)^{-1}(H) = f^{-1}(g^{-1}(H)).$$

Note the *reversal* in the order of the functions.

Restrictions of Functions

If $f: A \to B$ is a function and if $A_1 \subset A$, we can define a function $f_1: A_1 \to B$ by

$$f_1(x) := f(x) \qquad \text{for} \qquad x \in A_1.$$

The function f_1 is called the **restriction of f to A_1**. Sometimes it is denoted by $f_1 = f|A_1$.

It may seem strange to the reader that one would ever choose to throw away a part of a function, but there are some good reasons for doing so. For example, if $f : \mathbb{R} \to \mathbb{R}$ is the **squaring function:**

$$f(x) := x^2 \qquad \text{for} \qquad x \in \mathbb{R},$$

then f is not injective, so it cannot have an inverse function. However, if we restrict f to the set $A_1 := \{x : x \geq 0\}$, then the restriction $f|A_1$ is a bijection of A_1 onto A_1. Therefore, this restriction has an inverse function, which is the **positive square root function.** (Sketch a graph.)

Similarly, the trigonometric functions $S(x) := \sin x$ and $C(x) := \cos x$ are not injective on all of $\mathbb{R}$. However, by making suitable restrictions of these functions, one can obtain the **inverse sine** and the **inverse cosine** functions that the reader has undoubtedly already encountered.

Exercises for Section 1.1

1. Let $A := \{k : k \in \mathbb{N}, k \leq 20\}$, $B := \{3k - 1 : k \in \mathbb{N}\}$, and $C := \{2k + 1 : k \in \mathbb{N}\}$. Determine the sets:
   (a) $A \cap B \cap C$,
   (b) $(A \cap B) \backslash C$,
   (c) $(A \cap C) \backslash B$.
2. Draw diagrams to simplify and identify the following sets:
   (a) $A \backslash (B \backslash A)$,
   (b) $A \backslash (A \backslash B)$,
   (c) $A \cap (B \backslash A)$.
3. If A and B are sets, show that $A \subseteq B$ if and only if $A \cap B = A$.
4. Prove the second De Morgan Law [Theorem 1.1.4(b)].
5. Prove the Distributive Laws:
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

6. The **symmetric difference** of two sets A and B is the set D of all elements that belong to either A or B but not both. Represent D with a diagram.
   (a) Show that $D = (A \backslash B) \cup (B \backslash A)$.
   (b) Show that D is also given by $D = (A \cup B) \backslash (A \cap B)$.
7. For each $n \in \mathbb{N}$, let $A_n = \{(n+1)k : k \in \mathbb{N}\}$.
   (a) What is $A_1 \cap A_2$?
   (b) Determine the sets $\cup\{A_n : n \in \mathbb{N}\}$ and $\cap\{A_n : n \in \mathbb{N}\}$.
8. Draw diagrams in the plane of the Cartesian products $A \times B$ for the given sets A and B.
   (a) $A = \{x \in \mathbb{R} : 1 \le x \le 2 \text{ or } 3 \le x \le 4\}$, $B = \{x \in \mathbb{R} : x = 1 \text{ or } x = 2\}$.
   (b) $A = \{1, 2, 3\}$, $B = \{x \in \mathbb{R} : 1 \le x \le 3\}$.
9. Let $A := B := \{x \in \mathbb{R} : -1 \le x \le 1\}$ and consider the subset $C := \{(x, y) : x^2 + y^2 = 1\}$ of $A \times B$. Is this set a function? Explain.
10. Let $f(x) := 1/x^2$, $x \ne 0$, $x \in \mathbb{R}$.
    (a) Determine the direct image $f(E)$ where $E := \{x \in \mathbb{R} : 1 \le x \le 2\}$.
    (b) Determine the inverse image $f^{-1}(G)$ where $G := \{x \in \mathbb{R} : 1 \le x \le 4\}$.
11. Let $g(x) := x^2$ and $f(x) := x + 2$ for $x \in \mathbb{R}$, and let h be the composite function $h := g \circ f$.
    (a) Find the direct image $h(E)$ of $E := \{x \in \mathbb{R} : 0 \le x \le 1\}$.
    (b) Find the inverse image $h^{-1}(G)$ of $G := \{x \in \mathbb{R} : 0 \le x \le 4\}$.
12. Let $f(x) := x^2$ for $x \in \mathbb{R}$, and let $E := \{x \in \mathbb{R} : -1 \le x \le 0\}$ and $F := \{x \in \mathbb{R} : 0 \le x \le 1\}$. Show that $E \cap F = \{0\}$ and $f(E \cap F) = \{0\}$, while $f(E) = f(F) = \{y \in \mathbb{R} : 0 \le y \le 1\}$. Hence $f(E \cap F)$ is a proper subset of $f(E) \cap f(F)$. What happens if 0 is deleted from the sets E and F?
13. Let f and E, F be as in Exercise 12. Find the sets $E \backslash F$ and $f(E) \backslash f(f)$ and show that it is *not* true that $f(E \backslash F) \subseteq f(E) \backslash f(F)$.
14. Show that if $f : A \to B$ and E, F are subsets of A, then $f(E \cup F) = f(E) \cup f(F)$ and $f(E \cap F) \subseteq f(E) \cap f(F)$.
15. Show that if $f : A \to B$ and G, H are subsets of B, then $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$ and $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$.
16. Show that the function f defined by $f(x) := x/\sqrt{x^2+1}$, $x \in \mathbb{R}$, is a bijection of $\mathbb{R}$ onto $\{y : -1 < y < 1\}$.
17. For $a, b \in \mathbb{R}$ with $a < b$, find an explicit bijection of $A := \{x : a < x < b\}$ onto $B := \{y : 0 < y < 1\}$.
18. (a) Give an example of two functions f, g on $\mathbb{R}$ to $\mathbb{R}$ such that $f \ne g$, but such that $f \circ g = g \circ f$.
    (b) Give an example of three functions f, g, h on $\mathbb{R}$ such that $f \circ (g + h) \ne f \circ g + f \circ h$.
19. (a) Show that if $f : A \to B$ is injective and $E \subseteq A$, then $f^{-1}(f(E)) = E$. Give an example to show that equality need not hold if f is not injective.
    (b) Show that if $f : A \to B$ is surjective and $H \subseteq B$, then $f(f^{-1}(H)) = H$. Give an example to show that equality need not hold if f is not surjective.
20. (a) Suppose that f is an injection. Show that $f^{-1} \circ f(x) = x$ for all $x \in D(f)$ and that $f \circ f^{-1}(y) = y$ for all $y \in R(f)$.
    (b) If f is a bijection of A onto B, show that f^{-1} is a bijection of B onto A.
21. Prove that if $f : A \to B$ is bijective and $g : B \to C$ is bijective, then the composite $g \circ f$ is a bijective map of A onto C.
22. Let $f : A \to B$ and $g : B \to C$ be functions.
    (a) Show that if $g \circ f$ is injective, then f is injective.
    (b) Show that if $g \circ f$ is surjective, then g is surjective.
23. Prove Theorem 1.1.14.
24. Let f, g be functions such that $(g \circ f)(x) = x$ for all $x \in D(f)$ and $(f \circ g)(y) = y$ for all $y \in D(g)$. Prove that $g = f^{-1}$.

[This result can also be proved without using Mathematical Induction. If we let $s_n := 1 + r + r^2 + \cdots + r^n$, then $rs_n = r + r^2 + \cdots + r^{n+1}$, so that

$$(1 - r)s_n = s_n - rs_n = 1 - r^{n+1}.$$

If we divide by $1 - r$, we obtain the stated formula.]

(g) Careless use of the Principle of Mathematical Induction can lead to obviously absurd conclusions. The reader is invited to find the error in the "proof" of the following assertion.

Claim: If $n \in \mathbb{N}$ and if the maximum of the natural numbers p and q is n, then $p = q$.

"Proof." Let S be the subset of $\mathbb{N}$ for which the claim is true. Evidently, $1 \in S$ since if $p, q \in \mathbb{N}$ and their maximum is 1, then both equal 1 and so $p = q$. Now assume that $k \in S$ and that the maximum of p and q is $k + 1$. Then the maximum of $p - 1$ and $q - 1$ is k. But since $k \in S$, then $p - 1 = q - 1$ and therefore $p = q$. Thus, $k + 1 \in S$, and we conclude that the assertion is true for all $n \in \mathbb{N}$.

(h) There are statements that are true for *many* natural numbers but that are not true for *all* of them.

For example, the formula $p(n) := n^2 - n + 41$ gives a prime number for $n = 1, 2, \ldots, 40$. However, $p(41)$ is obviously divisible by 41, so it is not a prime number.

Another version of the Principle of Mathematical Induction is sometimes quite useful. It is called the "Principle of Strong Induction," even though it is in fact equivalent to 1.2.2.

1.2.5 Principle of Strong Induction *Let S be a subset of $\mathbb{N}$ such that*

(1″) $1 \in S$.
(2″) *For every $k \in \mathbb{N}$, if $\{1, 2, \ldots, k\} \subseteq S$, then $k + 1 \in S$.*

Then $S = \mathbb{N}$.

We will leave it to the reader to establish the equivalence of 1.2.2 and 1.2.5.

Exercises for Section 1.2

1. Prove that $1/1 \cdot 2 + 1/2 \cdot 3 + \cdots + 1/n(n+1) = n/(n+1)$ for all $n \in \mathbb{N}$.
2. Prove that $1^3 + 2^3 + \cdots + n^3 = \left[\frac{1}{2}n(n+1)\right]^2$ for all $n \in \mathbb{N}$.
3. Prove that $3 + 11 + \cdots + (8n - 5) = 4n^2 - n$ for all $n \in \mathbb{N}$.
4. Prove that $1^2 + 3^2 + \cdots + (2n-1)^2 = (4n^3 - n)/3$ for all $n \in \mathbb{N}$.
5. Prove that $1^2 - 2^2 + 3^2 + \cdots + (-1)^{n+1}n^2 = (-1)^{n+1}n(n+1)/2$ for all $n \in \mathbb{N}$.
6. Prove that $n^3 + 5n$ is divisible by 6 for all $n \in \mathbb{N}$.
7. Prove that $5^{2n} - 1$ is divisible by 8 for all $n \in \mathbb{N}$.
8. Prove that $5^n - 4n - 1$ is divisible by 16 for all $n \in \mathbb{N}$.
9. Prove that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for all $n \in \mathbb{N}$.
10. Conjecture a formula for the sum $1/1 \cdot 3 + 1/3 \cdot 5 + \cdots + 1/(2n-1)(2n+1)$, and prove your conjecture by using Mathematical Induction.
11. Conjecture a formula for the sum of the first n odd natural numbers $1 + 3 + \cdots + (2n - 1)$, and prove your formula by using Mathematical Induction.
12. Prove the Principle of Mathematical Induction 1.2.3 (second version).

13. Prove that $n < 2^n$ for all $n \in \mathbb{N}$.
14. Prove that $2^n < n!$ for all $n \geq 4$, $n \in \mathbb{N}$.
15. Prove that $2n - 3 \leq 2^{n-2}$ for all $n \geq 5$, $n \in \mathbb{N}$.
16. Find all natural numbers n such that $n^2 < 2^n$. Prove your assertion.
17. Find the largest natural number m such that $n^3 - n$ is divisible by m for all $n \in \mathbb{N}$. Prove your assertion.
18. Prove that $1/\sqrt{1} + 1/\sqrt{2} + \cdots + 1/\sqrt{n} > \sqrt{n}$ for all $n \in \mathbb{N}$, $n > 1$.
19. Let S be a subset of $\mathbb{N}$ such that (a) $2^k \in S$ for all $k \in \mathbb{N}$, and (b) if $k \in S$ and $k \geq 2$, then $k - 1 \in S$. Prove that $S = \mathbb{N}$.
20. Let the numbers x_n be defined as follows: $x_1 := 1$, $x_2 := 2$, and $x_{n+2} := \frac{1}{2}(x_{n+1} + x_n)$ for all $n \in \mathbb{N}$. Use the Principle of Strong Induction (1.2.5) to show that $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$.

Section 1.3 Finite and Infinite Sets

When we count the elements in a set, we say "one, two, three, . . . ," stopping when we have exhausted the set. From a mathematical perspective, what we are doing is defining a bijective mapping between the set and a portion of the set of natural numbers. If the set is such that the counting does not terminate, such as the set of natural numbers itself, then we describe the set as being infinite.

The notions of "finite" and "infinite" are extremely primitive, and it is very likely that the reader has never examined these notions very carefully. In this section we will define these terms precisely and establish a few basic results and state some other important results that seem obvious but whose proofs are a bit tricky. These proofs can be found in Appendix B and can be read later.

1.3.1 Definition **(a)** The empty set $\emptyset$ is said to have 0 **elements**.

(b) If $n \in \mathbb{N}$, a set S is said to have n **elements** if there exists a bijection from the set $\mathbb{N}_n := \{1, 2, \ldots, n\}$ onto S.

(c) A set S is said to be **finite** if it is either empty or it has n elements for some $n \in \mathbb{N}$.

(d) A set S is said to be **infinite** if it is not finite.

Since the inverse of a bijection is a bijection, it is easy to see that a set S has n elements if and only if there is a bijection from S onto the set $\{1, 2, \ldots, n\}$. Also, since the composition of two bijections is a bijection, we see that a set S_1 has n elements if and only if there is a bijection from S_1 onto another set S_2 that has n elements. Further, a set T_1 is finite if and only if there is a bijection from T_1 onto another set T_2 that is finite.

It is now necessary to establish some basic properties of finite sets to be sure that the definitions do not lead to conclusions that conflict with our experience of counting. From the definitions, it is not entirely clear that a finite set might not have n elements for *more than one* value of n. Also it is conceivably possible that the set $\mathbb{N} := \{1, 2, 3, \ldots\}$ might be a finite set according to this definition. The reader will be relieved that these possibilities do not occur, as the next two theorems state. The proofs of these assertions, which use the fundamental properties of $\mathbb{N}$ described in Section 1.2, are given in Appendix B.

1.3.2 Uniqueness Theorem *If S is a finite set, then the number of elements in S is a unique number in $\mathbb{N}$.*

Exercises for Section 1.3

1. Prove that a nonempty set T_1 is finite if and only if there is a bijection from T_1 onto a finite set T_2.
2. Prove parts (b) and (c) of Theorem 1.3.4.
3. Let $S := \{1, 2\}$ and $T := \{a, b, c\}$.
   (a) Determine the number of different injections from S into T.
   (b) Determine the number of different surjections from T onto S.
4. Exhibit a bijection between $\mathbb{N}$ and the set of all odd integers greater than 13.
5. Give an explicit definition of the bijection f from $\mathbb{N}$ onto $\mathbb{Z}$ described in Example 1.3.7(b).
6. Exhibit a bijection between $\mathbb{N}$ and a proper subset of itself.
7. Prove that a set T_1 is denumerable if and only if there is a bijection from T_1 onto a denumerable set T_2.
8. Give an example of a countable collection of finite sets whose union is not finite.
9. Prove in detail that if S and T are denumerable, then $S \cup T$ is denumerable.
10. (a) If (m, n) is the 6th point down the 9th diagonal of the array in Figure 1.3.1, calculate its number according to the counting method given for Theorem 1.3.8.
    (b) Given that $h(m, 3) = 19$, find m.
11. Determine the number of elements in $\mathcal{P}(S)$, the collection of all subsets of S, for each of the following sets:
    (a) $S := \{1, 2\}$,
    (b) $S := \{1, 2, 3\}$,
    (c) $S := \{1, 2, 3, 4\}$.
    Be sure to include the empty set and the set S itself in $\mathcal{P}(S)$.
12. Use Mathematical Induction to prove that if the set S has n elements, then $\mathcal{P}(S)$ has 2^n elements.
13. Prove that the collection $\mathcal{F}(\mathbb{N})$ of all *finite* subsets of $\mathbb{N}$ is countable.

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Therefore (2) holds (with strict inequality) when $a \neq b$. Moreover, if $a = b(> 0)$, then both sides of (2) equal a, so (2) becomes an equality. This proves that (2) holds for $a > 0, b > 0$.

On the other hand, suppose that $a > 0$, $b > 0$ and that $\sqrt{ab} = \frac{1}{2}(a+b)$. Then, squaring both sides and multiplying by 4, we obtain

$$4ab = (a+b)^2 = a^2 + 2ab + b^2,$$

whence it follows that

$$0 = a^2 - 2ab + b^2 = (a-b)^2.$$

But this equality implies that $a = b$. (Why?) Thus, equality in (2) implies that $a = b$.

Remark The general Arithmetic-Geometric Mean Inequality for the positive real numbers $a_1, a_2, \ldots, a_n$ is

$$(3) \qquad (a_1 a_2 \cdots a_n)^{1/n} \le \frac{a_1 + a_2 + \cdots + a_n}{n}$$

with equality occurring if and only if $a_1 = a_2 = \cdots = a_n$. It is possible to prove this more general statement using Mathematical Induction, but the proof is somewhat intricate. A more elegant proof that uses properties of the exponential function is indicated in Exercise 8.3.9 in Chapter 8.

(c) Bernoulli's Inequality. If $x > -1$, then

$$(4) \qquad (1+x)^n \ge 1 + nx \qquad \text{for all} \quad n \in \mathbb{N}$$

The proof uses Mathematical Induction. The case $n = 1$ yields equality, so the assertion is valid in this case. Next, we assume the validity of the inequality (4) for $k \in \mathbb{N}$ and will deduce it for $k + 1$. Indeed, the assumptions that $(1+x)^k \ge 1 + kx$ and that $1 + x > 0$ imply (why?) that

$$\begin{aligned}(1+x)^{k+1} &= (1+x)^k \cdot (1+x) \\ &\ge (1+kx) \cdot (1+x) = 1 + (k+1)x + kx^2 \\ &\ge 1 + (k+1)x.\end{aligned}$$

Thus, inequality (4) holds for $n = k + 1$. Therefore, (4) holds for all $n \in \mathbb{N}$. □

Exercises for Section 2.1

1. If $a, b \in \mathbb{R}$, prove the following.
   (a) If $a + b = 0$, then $b = -a$,
   (b) $-(-a) = a$,
   (c) $(-1)a = -a$,
   (d) $(-1)(-1) = 1$.

2. Prove that if $a, b \in \mathbb{R}$, then
   (a) $-(a+b) = (-a) + (-b)$,
   (b) $(-a) \cdot (-b) = a \cdot b$,
   (c) $1/(-a) = -(1/a)$,
   (d) $-(a/b) = (-a)/b$ if $b \neq 0$.

3. Solve the following equations, justifying each step by referring to an appropriate property or theorem.
   (a) $2x + 5 = 8$,
   (b) $x^2 = 2x$,
   (c) $x^2 - 1 = 3$,
   (d) $(x-1)(x+2) = 0$.

4. If $a \in \mathbb{R}$ satisfies $a \cdot a = a$, prove that either $a = 0$ or $a = 1$.

5. If $a \neq 0$ and $b \neq 0$, show that $1/(ab) = (1/a)(1/b)$.

6. Use the argument in the proof of Theorem 2.1.4 to show that there does not exist a rational number s such that $s^2 = 6$.

7. Modify the proof of Theorem 2.1.4 to show that there does not exist a rational number t such that $t^2 = 3$.

8. (a) Show that if x, y are rational numbers, then $x + y$ and xy are rational numbers.
   (b) Prove that if x is a rational number and y is an irrational number, then $x + y$ is an irrational number. If, in addition, $x \neq 0$, then show that xy is an irrational number.

9. Let $K := \{s + t\sqrt{2} : s, t \in \mathbb{Q}\}$. Show that K satisfies the following:
   (a) If $x_1, x_2 \in K$, then $x_1 + x_2 \in K$ and $x_1 x_2 \in K$.
   (b) If $x \neq 0$ and $x \in K$, then $1/x \in K$.
   (Thus the set K is a *subfield* of $\mathbb{R}$. With the order inherited from $\mathbb{R}$, the set K is an ordered field that lies between $\mathbb{Q}$ and $\mathbb{R}$.)

10. (a) If $a < b$ and $c \leq d$, prove that $a + c < b + d$.
    (b) If $0 < a < b$ and $0 \leq c \leq d$, prove that $0 \leq ac \leq bd$.

11. (a) Show that if $a > 0$, then $1/a > 0$ and $1/(1/a) = a$.
    (b) Show that if $a < b$, then $a < \frac{1}{2}(a + b) < b$.

12. Let a, b, c, d be numbers satisfying $0 < a < b$ and $c < d < 0$. Give an example where $ac < bd$, and one where $bd < ac$.

13. If $a, b \in \mathbb{R}$, show that $a^2 + b^2 = 0$ if and only if $a = 0$ and $b = 0$.

14. If $0 \leq a < b$, show that $a^2 \leq ab < b^2$. Show by example that it does *not* follow that $a^2 < ab < b^2$.

15. If $0 < a < b$, show that (a) $a < \sqrt{ab} < b$, and (b) $1/b < 1/a$.

16. Find all real numbers x that satisfy the following inequalities.
    (a) $x^2 > 3x + 4$, (b) $1 < x^2 < 4$,
    (c) $1/x < x$, (d) $1/x < x^2$.

17. Prove the following form of Theorem 2.1.9: If $a \in \mathbb{R}$ is such that $0 \leq a \leq \varepsilon$ for every $\varepsilon > 0$, then $a = 0$.

18. Let $a, b \in \mathbb{R}$, and suppose that for every $\varepsilon > 0$ we have $a \leq b + \varepsilon$. Show that $a \leq b$.

19. Prove that $\left[\frac{1}{2}(a + b)\right]^2 \leq \frac{1}{2}(a^2 + b^2)$ for all $a, b \in \mathbb{R}$. Show that equality holds if and only if $a = b$.

20. (a) If $0 < c < 1$, show that $0 < c^2 < c < 1$.
    (b) If $1 < c$, show that $1 < c < c^2$.

21. (a) Prove there is no $n \in \mathbb{N}$ such that $0 < n < 1$. (Use the Well-Ordering Property of $\mathbb{N}$.)
    (b) Prove that no natural number can be both even and odd.

22. (a) If $c > 1$, show that $c^n \geq c$ for all $n \in \mathbb{N}$, and that $c^n > c$ for $n > 1$.
    (b) If $0 < c < 1$, show that $c^n \leq c$ for all $n \in \mathbb{N}$, and that $c^n < c$ for $n > 1$.

23. If $a > 0, b > 0$, and $n \in \mathbb{N}$, show that $a < b$ if and only if $a^n < b^n$. [*Hint:* Use Mathematical Induction.]

24. (a) If $c > 1$ and $m, n \in \mathbb{N}$, show that $c^m > c^n$ if and only if $m > n$.
    (b) If $0 < c < 1$ and $m, n \in \mathbb{N}$, show that $c^m < c^n$ if and only if $m > n$.

25. Assuming the existence of roots, show that if $c > 1$, then $c^{1/m} < c^{1/n}$ if and only if $m > n$.

26. Use Mathematical Induction to show that if $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$, then $a^{m+n} = a^m a^n$ and $(a^m) = a^{mn}$.

Later we will need precise language to discuss the notion of one real number being "close to" another. If a is a given real number, then saying that a real number x is "close to" a should mean that the distance $|x - a|$ between them is "small." A context in which this idea can be discussed is provided by the terminology of neighborhoods, which we now define.

2.2.7 Definition Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Then the **ε-neighborhood** of a is the set $V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\}$.

For $a \in \mathbb{R}$, the statement that x belongs to $V_\varepsilon(a)$ is equivalent to either of the statements (see Figure 2.2.4)

$$-\varepsilon < x - a < \varepsilon \iff a - \varepsilon < x < a + \varepsilon.$$

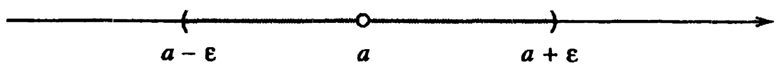

Figure 2.2.4 An ε-neighborhood of a

2.2.8 Theorem *Let $a \in \mathbb{R}$. If x belongs to the neighborhood $V_\varepsilon(a)$ for every $\varepsilon > 0$, then $x = a$.*

Proof. If a particular x satisfies $|x - a| < \varepsilon$ for every $\varepsilon > 0$, then it follows from 2.1.9 that $|x - a| = 0$, and hence $x = a$. Q.E.D.

2.2.9 Examples **(a)** Let $U := \{x : 0 < x < 1\}$. If $a \in U$, then let ε be the smaller of the two numbers a and $1 - a$. Then it is an exercise to show that $V_\varepsilon(a)$ is contained in U. Thus each element of U has some ε-neighborhood of it contained in U.

(b) If $I := \{x : 0 \le x \le 1\}$, then for any $\varepsilon > 0$, the ε-neighborhood $V_\varepsilon(0)$ of 0 contains points not in I, and so $V_\varepsilon(0)$ is not contained in I. For example, the number $x_\varepsilon := -\varepsilon/2$ is in $V_\varepsilon(0)$ but not in I.

(c) If $|x - a| < \varepsilon$ and $|y - b| < \varepsilon$, then the Triangle Inequality implies that

$$\begin{aligned} |(x + y) - (a + b)| &= |(x - a) + (y - b)| \\ &\le |x - a| + |y - b| < 2\varepsilon. \end{aligned}$$

Thus if x, y belong to the ε-neighborhoods of a, b, respectively, then $x + y$ belongs to the 2ε-neighborhood of $a + b$ (but not necessarily to the ε-neighborhood of $a + b$). □

Exercises for Section 2.2

1. If $a, b \in \mathbb{R}$ and $b \neq 0$, show that:
   (a) $|a| = \sqrt{a^2}$, (b) $|a/b| = |a|/|b|$.
2. If $a, b \in \mathbb{R}$, show that $|a + b| = |a| + |b|$ if and only if $ab \ge 0$.
3. If $x, y, z \in \mathbb{R}$ and $x \le z$, show that $x \le y \le z$ if and only if $|x - y| + |y - z| = |x - z|$. Interpret this geometrically.

4. Show that $|x - a| < \varepsilon$ if and only if $a - \varepsilon < x < a + \varepsilon$.
5. If $a < x < b$ and $a < y < b$, show that $|x - y| < b - a$. Interpret this geometrically.
6. Find all $x \in \mathbb{R}$ that satisfy the following inequalities:
   (a) $|4x - 5| \le 13$, (b) $|x^2 - 1| \le 3$.
7. Find all $x \in \mathbb{R}$ that satisfy the equation $|x + 1| + |x - 2| = 7$.
8. Find all values of x that satisfy the following equations:
   (a) $x + 1 = |2x - 1|$, (b) $2x - 1 = |x - 5|$.
9. Find all values of x that satisfy the following inequalities. Sketch graphs.
   (a) $|x - 2| \le x + 1$, (b) $3|x| \le 2 - x$.
10. Find all $x \in \mathbb{R}$ that satisfy the following inequalities.
    (a) $|x - 1| > |x + 1|$, (b) $|x| + |x + 1| < 2$.
11. Sketch the graph of the equation $y = |x| - |x - 1|$.
12. Find all $x \in \mathbb{R}$ that satisfy the inequality $4 < |x + 2| + |x - 1| < 5$.
13. Find all $x \in \mathbb{R}$ that satisfy both $|2x - 3| < 5$ and $|x + 1| > 2$ simultaneously.
14. Determine and sketch the set of pairs (x, y) in $\mathbb{R} \times \mathbb{R}$ that satisfy:
    (a) $|x| = |y|$, (b) $|x| + |y| = 1$,
    (c) $|xy| = 2$, (d) $|x| - |y| = 2$.
15. Determine and sketch the set of pairs (x, y) in $\mathbb{R} \times \mathbb{R}$ that satisfy:
    (a) $|x| \le |y|$, (b) $|x| + |y| \le 1$,
    (c) $|xy| \le 2$, (d) $|x| - |y| \ge 2$.
16. Let $\varepsilon > 0$ and $\delta > 0$, and $a \in \mathbb{R}$. Show that $V_\varepsilon(a) \cap V_\delta(a)$ and $V_\varepsilon(a) \cup V_\delta(a)$ are γ-neighborhoods of a for appropriate values of γ.
17. Show that if $a, b \in \mathbb{R}$, and $a \ne b$, then there exist ε-neighborhoods U of a and V of b such that $U \cap V = \emptyset$.
18. Show that if $a, b \in \mathbb{R}$ then
    (a) $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$ and $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$.
    (b) $\min\{a, b, c\} = \min\{\min\{a, b\}, c\}$.
19. Show that if $a, b, c \in \mathbb{R}$, then the "middle number" is $\text{mid}\{a, b, c\} = \min\{\max\{a, b\}, \max\{b, c\}, \max\{c, a\}\}$.

Section 2.3 The Completeness Property of $\mathbb{R}$

Thus far, we have discussed the algebraic properties and the order properties of the real number system $\mathbb{R}$. In this section we shall present one more property of $\mathbb{R}$ that is often called the "Completeness Property." The system $\mathbb{Q}$ of rational numbers also has the algebraic and order properties described in the preceding sections, but we have seen that $\sqrt{2}$ *cannot* be represented as a rational number; therefore $\sqrt{2}$ does not belong to $\mathbb{Q}$. This observation shows the necessity of an additional property to characterize the real number system. This additional property, the Completeness (or the Supremum) Property, is an essential property of $\mathbb{R}$, and we will say that $\mathbb{R}$ is a *complete ordered field*. It is this special property that permits us to define and develop the various limiting procedures that will be discussed in the chapters that follow.

There are several different ways to describe the Completeness Property. We choose to give what is probably the most efficient approach by assuming that each nonempty bounded subset of $\mathbb{R}$ has a supremum.

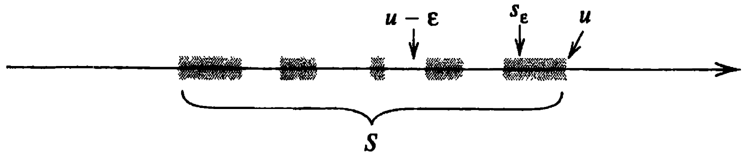

Figure 2.3.2 $u = \sup S$

2.3.5 Examples **(a)** If a nonempty set S_1 has a finite number of elements, then it can be shown that S_1 has a largest element u and a least element w. Then $u = \sup S_1$ and $w = \inf S_1$, and they are both members of S_1. (This is clear if S_1 has only one element, and it can be proved by induction on the number of elements in S_1; see Exercises 12 and 13.)

(b) The set $S_2 := \{x : 0 \leq x \leq 1\}$ clearly has 1 for an upper bound. We prove that 1 is its supremum as follows. If $v < 1$, there exists an element $s' \in S_2$ such that $v < s'$. (Name one such element s'.) Therefore v is not an upper bound of S_2 and, since v is an arbitrary number $v < 1$, we conclude that $\sup S_2 = 1$. It is similarly shown that $\inf S_2 = 0$. Note that both the supremum and the infimum of S_2 are contained in S_2.

(c) The set $S_3 := \{x : 0 < x < 1\}$ clearly has 1 for an upper bound. Using the same argument as given in (b), we see that $\sup S_3 = 1$. In this case, the set S_3 does *not* contain its supremum. Similarly, $\inf S_3 = 0$ is not contained in S_3. □

The Completeness Property of ℝ

It is not possible to prove on the basis of the field and order properties of ℝ that were discussed in Section 2.1 that every nonempty subset of ℝ that is bounded above has a supremum in ℝ. However, it is a deep and fundamental property of the real number system that this is indeed the case. We will make frequent and essential use of this property, especially in our discussion of limiting processes. The following statement concerning the existence of suprema is our final assumption about ℝ. Thus, we say that ℝ is a *complete ordered field.*

2.3.6 The Completeness Property of ℝ *Every nonempty set of real numbers that has an upper bound also has a supremum in ℝ.*

This property is also called the **Supremum Property** of ℝ. The analogous property for infima can be deduced from the Completeness Property as follows. Suppose that S is a nonempty subset of ℝ that is bounded below. Then the nonempty set $\overline{S} := \{-s : s \in S\}$ is bounded above, and the Supremum Property implies that $u := \sup \overline{S}$ exists in ℝ. The reader should verify in detail that $-u$ is the infimum of S.

Exercises for Section 2.3

1. Let $S_1 := \{x \in \mathbb{R} : x \geq 0\}$. Show in detail that the set S_1 has lower bounds, but no upper bounds. Show that $\inf S_1 = 0$.
2. Let $S_2 := \{x \in \mathbb{R} : x > 0\}$. Does S_2 have lower bounds? Does S_2 have upper bounds? Does $\inf S_2$ exist? Does $\sup S_2$ exist? Prove your statements.
3. Let $S_3 = \{1/n : n \in \mathbb{N}\}$. Show that $\sup S_3 = 1$ and $\inf S_3 \geq 0$. (It will follow from the Archimedean Property in Section 2.4 that $\inf S_3 = 0$.)
4. Let $S_4 := \{1 - (-1)^n/n : n \in \mathbb{N}\}$. Find $\inf S_4$ and $\sup S_4$.

5. Find the infimum and supremum, if they exist, of each of the following sets.
   (a) $A := \{x \in \mathbb{R} : 2x + 5 > 0\}$, (b) $B := \{x \in \mathbb{R} : x + 2 \geq x^2\}$,
   (c) $C := \{x \in \mathbb{R} : x < 1/x\}$, (d) $D := \{x \in \mathbb{R} : x^2 - 2x - 5 < 0\}$.
6. Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Prove that $\inf S = -\sup\{-s : s \in S\}$.
7. If a set $S \subseteq \mathbb{R}$ contains one of its upper bounds, show that this upper bound is the supremum of S.
8. Let $S \subseteq \mathbb{R}$ be nonempty. Show that $u \in \mathbb{R}$ is an upper bound of S if and only if the conditions $t \in \mathbb{R}$ and $t > u$ imply that $t \notin S$.
9. Let $S \subseteq \mathbb{R}$ be nonempty. Show that if $u = \sup S$, then for every number $n \in \mathbb{N}$ the number $u - 1/n$ is not an upper bound of S, but the number $u + 1/n$ is an upper bound of S. (The converse is also true; see Exercise 2.4.3.)
10. Show that if A and B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set. Show that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.
11. Let S be a bounded set in $\mathbb{R}$ and let S_0 be a nonempty subset of S. Show that $\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S$.
12. Let $S \subseteq \mathbb{R}$ and suppose that $s^* := \sup S$ belongs to S. If $u \notin S$, show that $\sup(S \cup \{u\}) = \sup\{s^*, u\}$.
13. Show that a nonempty finite set $S \subseteq \mathbb{R}$ contains its supremum. [*Hint:* Use Mathematical Induction and the preceding exercise.]
14. Let S be a set that is bounded below. Prove that a lower bound w of S is the infimum of S if and only if for any $\varepsilon > 0$ there exists $t \in S$ such that $t < w + \varepsilon$.

Section 2.4 Applications of the Supremum Property

We will now discuss how to work with suprema and infima. We will also give some very important applications of these concepts to derive fundamental properties of $\mathbb{R}$. We begin with examples that illustrate useful techniques in applying the ideas of supremum and infimum.

2.4.1 Examples (a) It is an important fact that taking suprema and infima of sets is compatible with the algebraic properties of $\mathbb{R}$. As an example, we present here the compatibility of taking suprema and addition.

Let S be a nonempty subset of $\mathbb{R}$ that is bounded above, and let a be any number in $\mathbb{R}$. Define the set $a + S := \{a + s : s \in S\}$. We will prove that

$$\sup(a + S) = a + \sup S.$$

If we let $u := \sup S$, then $x \leq u$ for all $x \in S$, so that $a + x \leq a + u$. Therefore, $a + u$ is an upper bound for the set $a + S$; consequently, we have $\sup(a + S) \leq a + u$.

Now if v is *any* upper bound of the set $a + S$, then $a + x \leq v$ for all $x \in S$. Consequently $x \leq v - a$ for all $x \in S$, so that $v - a$ is an upper bound of S. Therefore, $u = \sup S \leq v - a$, which gives us $a + u \leq v$. Since v is any upper bound of $a + S$, we can replace v by $\sup(a + S)$ to get $a + u \leq \sup(a + S)$.

Combining these inequalities, we conclude that

$$\sup(a + S) = a + u = a + \sup S.$$

For similar relationships between the suprema and infima of sets and the operations of addition and multiplication, see the exercises.

theorem can be formulated to establish the existence of a unique **positive *n*th root** of a, denoted by $\sqrt[n]{a}$ or $a^{1/n}$, for each $n \in \mathbb{N}$.

Remark If in the proof of Theorem 2.4.7 we replace the set S by the set of rational numbers $T := \{r \in \mathbb{Q} : 0 \leq r, r^2 < 2\}$, the argument then gives the conclusion that $y := \sup T$ satisfies $y^2 = 2$. Since we have seen in Theorem 2.1.4 that y cannot be a rational number, it follows that the set T that consists of rational numbers does not have a supremum belonging to the set $\mathbb{Q}$. Thus the ordered field $\mathbb{Q}$ of rational numbers does *not* possess the Completeness Property.

Density of Rational Numbers in $\mathbb{R}$

We now know that there exists at least one irrational real number, namely $\sqrt{2}$. Actually there are "more" irrational numbers than rational numbers in the sense that the set of rational numbers is countable (as shown in Section 1.3), while the set of irrational numbers is uncountable (see Section 2.5). However, we next show that in spite of this apparent disparity, the set of rational numbers is "dense" in $\mathbb{R}$ in the sense that given any two real numbers there is a rational number between them (in fact, there are infinitely many such rational numbers).

2.4.8 The Density Theorem *If x and y are any real numbers with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.*

Proof. It is no loss of generality (why?) to assume that $x > 0$. Since $y - x > 0$, it follows from Corollary 2.4.5 that there exists $n \in \mathbb{N}$ such that $1/n < y - x$. Therefore, we have $nx + 1 < ny$. If we apply Corollary 2.4.6 to $nx > 0$, we obtain $m \in \mathbb{N}$ with $m - 1 \leq nx < m$. Therefore, $m \leq nx + 1 < ny$, whence $nx < m < ny$. Thus, the rational number $r := m/n$ satisfies $x < r < y$. Q.E.D.

To round out the discussion of the interlacing of rational and irrational numbers, we have the same "betweenness property" for the set of irrational numbers.

2.4.9 Corollary *If x and y are real numbers with $x < y$, then there exists an irrational number z such that $x < z < y$.*

Proof. If we apply the Density Theorem 2.4.8 to the real numbers $x/\sqrt{2}$ and $y/\sqrt{2}$, we obtain a rational number $r \neq 0$ (why?) such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}.$$

Then $z := r\sqrt{2}$ is irrational (why?) and satisfies $x < z < y$. Q.E.D.

Exercises for Section 2.4

1. Show that $\sup\{1 - 1/n : n \in \mathbb{N}\} = 1$.
2. If $S := \{1/n - 1/m : n, m \in \mathbb{N}\}$, find $\inf S$ and $\sup S$.
3. Let $S \subseteq \mathbb{R}$ be nonempty. Prove that if a number u in $\mathbb{R}$ has the properties: (i) for every $n \in \mathbb{N}$ the number $u - 1/n$ is not an upper bound of S, and (ii) for every number $n \in \mathbb{N}$ the number $u + 1/n$ is an upper bound of S, then $u = \sup S$. (This is the converse of Exercise 2.3.9.)

4. Let S be a nonempty bounded set in $\mathbb{R}$.
   (a) Let $a > 0$, and let $aS := \{as : s \in S\}$. Prove that

$$\inf(aS) = a \inf S, \quad \sup(aS) = a \sup S.$$

   (b) Let $b < 0$ and let $bS = \{bs : s \in S\}$. Prove that

$$\inf(bS) = b \sup S, \quad \sup(bS) = b \inf S.$$

5. Let S be a set of nonnegative real numbers that is bounded above and let $T := \{x^2 : x \in S\}$. Prove that if $u = \sup S$, then $u^2 = \sup T$. Give an example that shows the conclusion may be false if the restriction against negative numbers is removed.

6. Let X be a nonempty set and let $f : X \to \mathbb{R}$ have bounded range in $\mathbb{R}$. If $a \in \mathbb{R}$, show that Example 2.4.1(a) implies that

$$\sup\{a + f(x) : x \in X\} = a + \sup\{f(x) : x \in X\}.$$

   Show that we also have

$$\inf\{a + f(x) : x \in X\} = a + \inf\{f(x) : x \in X\}.$$

7. Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$.

8. Let X be a nonempty set, and let f and g be defined on X and have bounded ranges in $\mathbb{R}$. Show that

$$\sup\{f(x) + g(x) : x \in X\} \le \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

   and that

$$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \le \inf\{f(x) + g(x) : x \in X\}.$$

   Give examples to show that each of these inequalities can be either equalities or strict inequalities.

9. Let $X = Y := \{x \in \mathbb{R} : 0 < x < 1\}$. Define $h : X \times Y \to \mathbb{R}$ by $h(x, y) := 2x + y$.
   (a) For each $x \in X$, find $f(x) := \sup\{h(x, y) : y \in Y\}$; then find $\inf\{f(x) : x \in X\}$.
   (b) For each $y \in Y$, find $g(y) := \inf\{h(x, y) : x \in X\}$; then find $\sup\{g(y) : y \in Y\}$. Compare with the result found in part (a).

10. Perform the computations in (a) and (b) of the preceding exercise for the function $h : X \times Y \to \mathbb{R}$ defined by

$$h(x, y) := \begin{cases} 0 & \text{if } x < y, \\ 1 & \text{if } x \ge y. \end{cases}$$

11. Let X and Y be nonempty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be defined by

$$f(x) := \sup\{h(x, y) : y \in Y\}, \quad g(y) := \inf\{h(x, y) : x \in X\}.$$

   Prove that

$$\sup\{g(y) : y \in Y\} \le \inf\{f(x) : x \in X\}.$$

   We sometimes express this by writing

$$\sup_y \inf_x h(x, y) \le \inf_x \sup_y h(x, y).$$

   Note that Exercises 9 and 10 show that the inequality may be either an equality or a strict inequality.

12. Let X and Y be nonempty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $F : X \to \mathbb{R}$ and $G : Y \to \mathbb{R}$ be defined by

$$F(x) := \sup\{h(x,y) : y \in Y\}, \quad G(y) := \sup\{h(x,y) : x \in X\}.$$

Establish the **Principle of the Iterated Suprema:**

$$\sup\{h(x,y) : x \in X, y \in Y\} = \sup\{F(x) : x \in X\} = \sup\{G(y) : y \in Y\}$$

We sometimes express this in symbols by

$$\sup_{x,y} h(x,y) = \sup_x \sup_y h(x,y) = \sup_y \sup_x h(x,y).$$

13. Given any $x \in \mathbb{R}$, show that there exists a *unique* $n \in \mathbb{Z}$ such that $n - 1 \leq x < n$.
14. If $y > 0$, show that there exists $n \in \mathbb{N}$ such that $1/2^n < y$.
15. Modify the argument in Theorem 2.4.7 to show that there exists a positive real number y such that $y^2 = 3$.
16. Modify the argument in Theorem 2.4.7 to show that if $a > 0$, then there exists a positive real number z such that $z^2 = a$.
17. Modify the argument in Theorem 2.4.7 to show that there exists a positive real number u such that $u^3 = 2$.
18. Complete the proof of the Density Theorem 2.4.8 by removing the assumption that $x > 0$.
19. If $u > 0$ is any real number and $x < y$, show that there exists a rational number r such that $x < ru < y$. (Hence the set $\{ru : r \in \mathbb{Q}\}$ is dense in $\mathbb{R}$.)

Section 2.5 Intervals

The Order Relation on $\mathbb{R}$ determines a natural collection of subsets called "intervals." The notations and terminology for these special sets will be familiar from earlier courses. If $a, b \in \mathbb{R}$ satisfy $a < b$, then the **open interval** determined by a and b is the set

$$(a,b) := \{x \in \mathbb{R} : a < x < b\}.$$

The points a and b are called the **endpoints** of the interval; however, the endpoints are not included in an open interval. If both endpoints are adjoined to this open interval, then we obtain the **closed interval** determined by a and b; namely, the set

$$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}.$$

The two **half-open** (or **half-closed**) intervals determined by a and b are $[a, b)$, which includes the endpoint a, and $(a, b]$, which includes the endpoint b.

Each of these four intervals is bounded and has **length** defined by $b - a$. If $a = b$, the corresponding open interval is the empty set $(a,a) = \emptyset$, whereas the corresponding closed interval is the singleton set $[a,a] = \{a\}$.

There are five types of unbounded intervals for which the symbols ∞(or $+\infty$) and $-\infty$ are used as notational convenience in place of the endpoints. The **infinite open intervals** are the sets of the form

$$(a,\infty) := \{x \in \mathbb{R} : x > a\} \quad \text{and} \quad (-\infty,b) := \{x \in \mathbb{R} : x < b\}.$$

obtaining $10x = 73.1414\cdots$. We now multiply by a power of 10 to move one block to the left of the decimal point; here getting $1000x = 7314.1414\cdots$. We now subtract to obtain an integer; here getting $1000x - 10x = 7314 - 73 = 7241$, whence $x = 7241/990$, a rational number.

Cantor's Second Proof

We will now give Cantor's second proof of the uncountability of $\mathbb{R}$. This is the elegant "diagonal" argument based on decimal representations of real numbers.

2.5.5 Theorem *The unit interval* $[0, 1] := \{x \in \mathbb{R} : 0 \le x \le 1\}$ *is not countable.*

Proof. The proof is by contradiction. We will use the fact that every real number $x \in [0, 1]$ has a decimal representation $x = 0.b_1b_2b_3\cdots$, where $b_i = 0, 1, \ldots, 9$. Suppose that there is an enumeration $x_1, x_2, x_3\cdots$ of all numbers in [0,1], which we display as:

$$\begin{aligned} x_1 &= 0.b_{11}b_{12}b_{13}\cdots b_{1n}\cdots, \\ x_2 &= 0.b_{21}b_{22}b_{23}\cdots b_{2n}\cdots, \\ x_3 &= 0.b_{31}b_{32}b_{33}\cdots b_{3n}\cdots, \\ &\cdots \\ x_n &= 0.b_{n1}b_{n2}b_{n3}\cdots b_{nn}\cdots, \\ &\cdots \end{aligned}$$

We now define a real number $y := 0.y_1y_2y_3\cdots y_n\cdots$ by setting $y_1 := 2$ if $b_{11} \ge 5$ and $y_1 := 7$ if $b_{11} \le 4$; in general, we let

$$y_n := \begin{cases} 2 & \text{if } b_{nn} \ge 5, \\ 7 & \text{if } b_{nn} \le 4. \end{cases}$$

Then $y \in [0, 1]$. Note that the number y is not equal to any of the numbers with two decimal representations, since $y_n \ne 0, 9$ for all $n \in \mathbb{N}$. Further, since y and x_n differ in the nth decimal place, then $y \ne x_n$ for any $n \in \mathbb{N}$. Therefore, y is not included in the enumeration of [0,1], contradicting the hypothesis. Q.E.D.

Exercises for Section 2.5

1. If $I := [a, b]$ and $I' := [a', b']$ are closed intervals in $\mathbb{R}$, show that $I \subseteq I'$ if and only if $a' \le a$ and $b \le b'$.
2. If $S \subseteq \mathbb{R}$ is nonempty, show that S is bounded if and only if there exists a closed bounded interval I such that $S \subseteq I$.
3. If $S \subseteq \mathbb{R}$ is a nonempty bounded set, and $I_S := [\inf S, \sup S]$, show that $S \subseteq I_S$. Moreover, if J is any closed bounded interval containing S, show that $I_S \subseteq J$.
4. In the proof of Case (ii) of Theorem 2.5.1, explain why x, y exist in S.
5. Write out the details of the proof of Case (iv) in Theorem 2.5.1.
6. If $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ is a nested sequence of intervals and if $I_n = [a_n, b_n]$, show that $a_1 \le a_2 \le \cdots \le a_n \le \cdots$ and $b_1 \ge b_2 \ge \cdots \ge b_n \ge \cdots$.
7. Let $I_n := [0, 1/n]$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} I_n = \{0\}$.
8. Let $J_n := (0, 1/n)$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} J_n = \emptyset$.
9. Let $K_n := (n, \infty)$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} K_n = \emptyset$.

10. With the notation in the proofs of Theorems 2.5.2 and 2.5.3, show that we have $\eta \in \bigcap_{n=1}^{\infty} I_n$. Also show that $[\xi, \eta] = \bigcap_{n=1}^{\infty} I_n$.

11. Show that the intervals obtained from the inequalities in (2) form a nested sequence.

12. Give the two binary representations of $\frac{3}{8}$ and $\frac{7}{16}$.

13. (a) Give the first four digits in the binary representation of $\frac{1}{3}$.
    (b) Give the complete binary representation of $\frac{1}{3}$.

14. Show that if $a_k, b_k \in \{0, 1, \ldots, 9\}$ and if

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} = \frac{b_1}{10} + \frac{b_2}{10^2} + \cdots + \frac{b_m}{10^m} \neq 0,$$

then $n = m$ and $a_k = b_k$ for $k = 1, \ldots, n$.

15. Find the decimal representation of $-\frac{2}{7}$.

16. Express $\frac{1}{7}$ and $\frac{2}{19}$ as periodic decimals.

17. What rationals are represented by the periodic decimals $1.25137\cdots137\cdots$ and $35.14653\cdots653\cdots$?

CH3

If $c > 1$, then $c^{1/n} = 1 + d_n$ for some $d_n > 0$. Hence by Bernoulli's Inequality 2.1.13(c),

$$c = (1 + d_n)^n \geq 1 + nd_n \quad \text{for} \quad n \in \mathbb{N}.$$

Therefore we have $c - 1 \geq nd_n$, so that $d_n \leq (c-1)/n$. Consequently we have

$$|c^{1/n} - 1| = d_n \leq (c-1)\frac{1}{n} \quad \text{for} \quad n \in \mathbb{N}.$$

We now invoke Theorem 3.1.10 to infer that $\lim(c^{1/n}) = 1$ when $c > 1$.

Now suppose that $0 < c < 1$; then $c^{1/n} = 1/(1 + h_n)$ for some $h_n > 0$. Hence Bernoulli's Inequality implies that

$$c = \frac{1}{(1+h_n)^n} \leq \frac{1}{1+nh_n} < \frac{1}{nh_n},$$

from which it follows that $0 < h_n < 1/nc$ for $n \in \mathbb{N}$. Therefore we have

$$0 < 1 - c^{1/n} = \frac{h_n}{1+h_n} < h_n < \frac{1}{nc}$$

so that

$$|c^{1/n} - 1| < \left(\frac{1}{c}\right)\frac{1}{n} \quad \text{for} \quad n \in \mathbb{N}.$$

We now apply Theorem 3.1.10 to infer that $\lim(c^{1/n}) = 1$ when $0 < c < 1$.

(d) $\lim(n^{1/n}) = 1$

Since $n^{1/n} > 1$ for $n > 1$, we can write $n^{1/n} = 1 + k_n$ for some $k_n > 0$ when $n > 1$. Hence $n = (1 + k_n)^n$ for $n > 1$. By the Binomial Theorem, if $n > 1$ we have

$$n = 1 + nk_n + \tfrac{1}{2}n(n-1)k_n^2 + \cdots \geq 1 + \tfrac{1}{2}n(n-1)k_n^2,$$

whence it follows that

$$n - 1 \geq \tfrac{1}{2}n(n-1)k_n^2.$$

Hence $k_n^2 \leq 2/n$ for $n > 1$. If $\varepsilon > 0$ is given, it follows from the Archimedean Property that there exists a natural number N_ε such that $2/N_\varepsilon < \varepsilon^2$. It follows that if $n \geq \sup\{2, N_\varepsilon\}$ then $2/n < \varepsilon^2$, whence

$$0 < n^{1/n} - 1 = k_n \leq (2/n)^{1/2} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $\lim(n^{1/n}) = 1$. □

Exercises for Section 3.1

1. The sequence (x_n) is defined by the following formulas for the nth term. Write the first five terms in each case:

   (a) $x_n := 1 + (-1)^n$, (b) $x_n := (-1)^n/n$,

   (c) $x_n := \dfrac{1}{n(n+1)}$, (d) $x := \dfrac{1}{n^2+2}$.

2. The first few terms of a sequence (x_n) are given below. Assuming that the "natural pattern" indicated by these terms persists, give a formula for the nth term x_n.
   (a) 5, 7, 9, 11, . . . , (b) 1/2, −1/4, 1/8, −1/16, . . .,
   (c) 1/2, 2/3, 3/4, 4/5, . . . , (d) 1, 4, 9, 16, ...
3. List the first five terms of the following inductively defined sequences.
   (a) $x_1 := 1, \quad x_{n+1} := 3x_n + 1$,
   (b) $y_1 := 2, \quad y_{n+1} := \frac{1}{2}(y_n + 2/y_n)$,
   (c) $z_1 := 1, \quad z_2 := 2, \ z_{n+2} := (z_{n+1} + z_n)/(z_{n+1} - z_n)$,
   (d) $s_1 := 3, \quad s_2 := 5, \ s_{n+2} := s_n + s_{n+1}$.
4. For any $b \in \mathbb{R}$, prove that $\lim(b/n) = 0$.
5. Use the definition of the limit of a sequence to establish the following limits.

   (a) $\lim\left(\dfrac{n}{n^2+1}\right) = 0$, (b) $\lim\left(\dfrac{2n}{n+1}\right) = 2$,

   (c) $\lim\left(\dfrac{3n+1}{2n+5}\right) = \dfrac{3}{2}$, (d) $\lim\left(\dfrac{n^2-1}{2n^2+3}\right) = \dfrac{1}{2}$.
6. Show that

   (a) $\lim\left(\dfrac{1}{\sqrt{n+7}}\right) = 0$, (b) $\lim\left(\dfrac{2n}{n+2}\right) = 2$,

   (c) $\lim\left(\dfrac{\sqrt{n}}{n+1}\right) = 0$, (d) $\lim\left(\dfrac{(-1)^n n}{n^2+1}\right) = 0$.
7. Let $x_n := 1/\ln(n+1)$ for $n \in \mathbb{N}$.
   (a) Use the definition of limit to show that $\lim(x_n) = 0$.
   (b) Find a specific value of $K(\varepsilon)$ as required in the definition of limit for each of (i) $\varepsilon = 1/2$, and (ii) $\varepsilon = 1/10$.
8. Prove that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$. Give an example to show that the convergence of $(|x_n|)$ need not imply the convergence of (x_n).
9. Show that if $x_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim(x_n) = 0$, then $\lim(\sqrt{x_n}) = 0$.
10. Prove that if $\lim(x_n) = x$ and if $x > 0$, then there exists a natural number M such that $x_n > 0$ for all $n \geq M$.
11. Show that $\lim\left(\dfrac{1}{n} - \dfrac{1}{n+1}\right) = 0$.
12. Show that $\lim(\sqrt{n^2+1} - n) = 0$.
13. Show that $\lim(1/3^n) = 0$.
14. Let $b \in \mathbb{R}$ satisfy $0 < b < 1$. Show that $\lim(nb^n) = 0$. [*Hint*: Use the Binomial Theorem as in Example 3.1.11(d).]
15. Show that $\lim\left((2n)^{1/n}\right) = 1$.
16. Show that $\lim(n^2/n!) = 0$.
17. Show that $\lim(2^n/n!) = 0$. [*Hint*: If $n \geq 3$, then $0 < 2^n/n! \leq 2\left(\frac{2}{3}\right)^{n-2}$.]
18. If $\lim(x_n) = x > 0$, show that there exists a natural number K such that if $n \geq K$, then $\frac{1}{2}x < x_n < 2x$.

3.2.11 Theorem *Let* (x_n) *be a sequence of positive real numbers such that* $L := \lim(x_{n+1}/x_n)$ *exists. If* $L < 1$, *then* (x_n) *converges and* $\lim(x_n) = 0$.

Proof. By 3.2.4 it follows that $L \geq 0$. Let r be a number such that $L < r < 1$, and let $\varepsilon := r - L > 0$. There exists a number $K \in \mathbb{N}$ such that if $n \geq K$ then

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \varepsilon.$$

It follows from this (why?) that if $n \geq K$, then

$$\frac{x_{n+1}}{x_n} < L + \varepsilon = L + (r - L) = r.$$

Therefore, if $n \geq K$, we obtain

$$0 < x_{n+1} < x_n r < x_{n-1} r^2 < \cdots < x_K r^{n-K+1}.$$

If we set $C := x_K / r^K$, we see that $0 < x_{n+1} < C r^{n+1}$ for all $n \geq K$. Since $0 < r < 1$, it follows from 3.1.11(b) that $\lim(r^n) = 0$ and therefore from Theorem 3.1.10 that $\lim(x_n) = 0$. Q.E.D.

As an illustration of the utility of the preceding theorem, consider the sequence (x_n) given by $x_n := n/2^n$. We have

$$\frac{x_{n+1}}{x_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2}\left(1 + \frac{1}{n}\right),$$

so that $\lim(x_{n+1}/x_n) = \frac{1}{2}$. Since $\frac{1}{2} < 1$, it follows from Theorem 3.2.11 that $\lim(n/2^n) = 0$.

Exercises for Section 3.2

1. For x_n given by the following formulas, establish either the convergence or the divergence of the sequence $X = (x_n)$.
   (a) $x_n := \dfrac{n}{n+1}$, (b) $x_n := \dfrac{(-1)^n n}{n+1}$,
   (c) $x_n := \dfrac{n^2}{n+1}$, (d) $x_n := \dfrac{2n^2+3}{n^2+1}$.
2. Give an example of two divergent sequences X and Y such that:
   (a) their sum $X + Y$ converges, (b) their product XY converges.
3. Show that if X and Y are sequences such that X and $X + Y$ are convergent, then Y is convergent.
4. Show that if X and Y are sequences such that X converges to $x \neq 0$ and XY converges, then Y converges.
5. Show that the following sequences are not convergent.
   (a) (2^n), (b) $((-1)^n n^2)$.
6. Find the limits of the following sequences:
   (a) $\lim\left((2 + 1/n)^2\right)$, (b) $\lim\left(\dfrac{(-1)^n}{n+2}\right)$,
   (c) $\lim\left(\dfrac{\sqrt{n}-1}{\sqrt{n}+1}\right)$, (d) $\lim\left(\dfrac{n+1}{n\sqrt{n}}\right)$.

7. If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Theorem 3.2.3 *cannot* be used.

8. Explain why the result in equation (3) before Theorem 3.2.4 *cannot* be used to evaluate the limit of the sequence $((1 + 1/n)^n)$.

9. Let $y_n := \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that $(\sqrt{n}\, y_n)$ converges. Find the limit.

10. Determine the limits of the following sequences.
    (a) $(\sqrt{4n^2 + n} - 2n)$, (b) $(\sqrt{n^2 + 5n} - n)$.

11. Determine the following limits.
    (a) $\lim\left((3\sqrt{n})^{1/2n}\right)$, (b) $\lim\left((n+1)^{1/\ln(n+1)}\right)$.

12. If $0 < a < b$, determine $\lim\left(\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}\right)$.

13. If $a > 0, b > 0$, show that $\lim\left(\sqrt{(n+a)(n+b)} - n\right) = (a+b)/2$.

14. Use the Squeeze Theorem 3.2.7 to determine the limits of the following,
    (a) (n^{1/n^2}), (b) $\left((n!)^{1/n^2}\right)$.

15. Show that if $z_n := (a^n + b^n)^{1/n}$ where $0 < a < b$, then $\lim(z_n) = b$.

16. Apply Theorem 3.2.11 to the following sequences, where a, b satisfy $0 < a < 1, b > 1$.
    (a) (a^n), (b) $(b^n/2^n)$,
    (c) (n/b^n), (d) $(2^{3n}/3^{2n})$.

17. (a) Give an example of a convergent sequence (x_n) of positive numbers with $\lim(x_{n+1}/x_n) = 1$.
    (b) Give an example of a divergent sequence with this property. (Thus, this property cannot be used as a test for convergence.)

18. Let $X = (x_n)$ be a sequence of positive real numbers such that $\lim(x_{n+1}/x_n) = L > 1$. Show that X is not a bounded sequence and hence is not convergent.

19. Discuss the convergence of the following sequences, where a, b satisfy $0 < a < 1, b > 1$.
    (a) $(n^2 a^n)$, (b) (b^n/n^2),
    (c) $(b^n/n!)$, (d) $(n!/n^n)$.

20. Let (x_n) be a sequence of positive real numbers such that $\lim(x_n^{1/n}) = L < 1$. Show that there exists a number r with $0 < r < 1$ such that $0 < x_n < r^n$ for all sufficiently large $n \in \mathbb{N}$. Use this to show that $\lim(x_n) = 0$.

21. (a) Give an example of a convergent sequence (x_n) of positive numbers with $\lim(x_n^{1/n}) = 1$.
    (b) Give an example of a divergent sequence (x_n) of positive numbers with $\lim(x_n^{1/n}) = 1$. (Thus, this property cannot be used as a test for convergence.)

22. Suppose that (x_n) is a convergent sequence and (y_n) is such that for any $\varepsilon > 0$ there exists M such that $|x_n - y_n| < \varepsilon$ for all $n \geq M$. Does it follow that (y_n) is convergent?

23. Show that if (x_n) and (y_n) are convergent sequences, then the sequences (u_n) and (v_n) defined by $u_n := \max\{x_n, y_n\}$ and $v_n := \min\{x_n, y_n\}$ are also convergent. (See Exercise 2.2.18.)

24. Show that if (x_n), (y_n), (z_n) are convergent sequences, then the sequence (w_n) defined by $w_n := \operatorname{mid}\{x_n, y_n, z_n\}$ is also convergent. (See Exercise 2.2.19.)

Section 3.3 Monotone Sequences

Until now, we have obtained several methods of showing that a sequence $X = (x_n)$ of real numbers is convergent:

In 1741, Euler accepted an offer from Frederick the Great to join the Berlin Academy, where he stayed for 25 years. During this period he wrote landmark books on a relatively new subject called calculus and a steady stream of papers on mathematics and science. In response to a request for instruction in science from the Princess of Anhalt-Dessau, he wrote her nearly 200 letters on science that later became famous in a book titled *Letters to a German Princess*. When Euler lost vision in one eye, Frederick thereafter referred to him as his mathematical "cyclops."

In 1766, he happily returned to Russia at the invitation of Catherine the Great. His eyesight continued to deteriorate and in 1771 he became totally blind following an eye operation. Incredibly, his blindness made little impact on his mathematics output, for he wrote several books and over 400 papers while blind. He remained active until the day of his death.

Euler's productivity was remarkable. He wrote textbooks on physics, algebra, calculus, real and complex analysis, and differential geometry. He also wrote hundreds of papers, many winning prizes. A current edition of his collected works consists of 74 volumes.

Exercises for Section 3.3

1. Let $x_1 := 8$ and $x_{n+1} := \frac{1}{2}x_n + 2$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Find the limit.
2. Let $x_1 > 1$ and $x_{n+1} := 2 - 1/x_n$ for $n \in \mathbb{N}$. show that (x_n) is bounded and monotone. Find the limit.
3. Let $x_1 \geq 2$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.
4. Let $x_1 := 1$ and $x_{n+1} := \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that (x_n) converges and find the limit.
5. Let $y_1 := \sqrt{p}$, where $p > 0$, and $y_{n+1} := \sqrt{p + y_n}$ for $n \in \mathbb{N}$. Show that (y_n) converges and find the limit. [*Hint:* One upper bound is $1 + 2\sqrt{p}$.]
6. Let $a > 0$ and let $z_1 > 0$. Define $z_{n+1} := \sqrt{a + z_n}$ for $n \in \mathbb{N}$. Show that (z_n) converges and find the limit.
7. Let $x_1 := a > 0$ and $x_{n+1} := x_n + 1/x_n$ for $n \in \mathbb{N}$. Determine whether (x_n) converges or diverges.
8. Let (a_n) be an increasing sequence, (b_n) be a decreasing sequence, and assume that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Show that $\lim(a_n) \leq \lim(b_n)$, and thereby deduce the Nested Intervals Property 2.5.2 from the Monotone Convergence Theorem 3.3.2.
9. Let A be an infinite subset of $\mathbb{R}$ that is bounded above and let $u := \sup A$. Show there exists an increasing sequence (x_n) with $x_n \in A$ for all $n \in \mathbb{N}$ such that $u = \lim(x_n)$.
10. Establish the convergence or the divergence of the sequence (y_n), where
$$y_n := \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \quad \text{for} \quad n \in \mathbb{N}.$$
11. Let $x_n := 1/1^2 + 1/2^2 + \cdots + 1/n^2$ for each $n \in \mathbb{N}$. Prove that (x_n) is increasing and bounded, and hence converges. [*Hint:* Note that if $k \geq 2$, then $1/k^2 \leq 1/k(k-1) = 1/(k-1) - 1/k$.]
12. Establish the convergence and find the limits of the following sequences.
    (a) $((1 + 1/n)^{n+1})$, (b) $((1 + 1/n)^{2n})$,
    (c) $\left(\left(1 + \dfrac{1}{n+1}\right)^n\right)$, (d) $((1 - 1/n)^n)$.
13. Use the method in Example 3.3.5 to calculate $\sqrt{2}$, correct to within 4 decimals.
14. Use the method in Example 3.3.5 to calculate $\sqrt{5}$, correct to within 5 decimals.
15. Calculate the number e_n in Example 3.3.6 for $n = 2, 4, 8, 16$.
16. Use a calculator to compute e_n for $n = 50$, $n = 100$, and $n = 1000$.

(d) implies (a). Let $w = \sup S$. If $\varepsilon > 0$ is given, then there are at most finitely many n with $w + \varepsilon < x_n$. Therefore $w + \varepsilon$ belongs to V and $\limsup (x_n) \le w + \varepsilon$. On the other hand, there exists a subsequence of (x_n) converging to some number larger than $w - \varepsilon$, so that $w - \varepsilon$ is not in V, and hence $w - \varepsilon \le \limsup (x_n)$. Since $\varepsilon > 0$ is arbitrary, we conclude that $w = \limsup (x_n)$. Q.E.D.

As an instructive exercise, the reader should formulate the corresponding theorem for the limit inferior of a bounded sequence of real numbers.

3.4.12 Theorem A bounded sequence (x_n) is convergent if and only if $\limsup (x_n) = \liminf (x_n)$.

We leave the proof as an exercise. Other basic properties can also be found in the exercises.

Exercises for Section 3.4

1. Give an example of an unbounded sequence that has a convergent subsequence.
2. Use the method of Example 3.4.3(b) to show that if $0 < c < 1$, then $\lim(c^{1/n}) = 1$.
3. Let (f_n) be the Fibonacci sequence of Example 3.1.2(d), and let $x_n := f_{n+1}/f_n$. Given that $\lim(x_n) = L$ exists, determine the value of L.
4. Show that the following sequences are divergent.
   (a) $\left(1 - (-1)^n + 1/n\right)$, (b) $(\sin n\pi/4)$.
5. Let $X = (x_n)$ and $Y = (y_n)$ be given sequences, and let the "shuffled" sequence $Z = (z_n)$ be defined by $z_1 := x_1, z_2 := y_1, \ldots, z_{2n-1} := x_n, z_{2n} := y_n, \ldots$. Show that Z is convergent if and only if both X and Y are convergent and $\lim X = \lim Y$.
6. Let $x_n := n^{1/n}$ for $n \in \mathbb{N}$.
   (a) Show that $x_{n+1} < x_n$ if and only if $(1 + 1/n)^n < n$, and infer that the inequality is valid for $n \ge 3$. (See Example 3.3.6.) Conclude that (x_n) is ultimately decreasing and that $x := \lim(x_n)$ exists.
   (b) Use the fact that the subsequence (x_{2n}) also converges to x to conclude that $x = 1$.
7. Establish the convergence and find the limits of the following sequences:
   (a) $\left((1 + 1/n^2)^{n^2}\right)$, (b) $\left((1 + 1/2n)^n\right)$,
   (c) $\left((1 + 1/n^2)^{2n^2}\right)$, (d) $\left((1 + 2/n)^n\right)$.
8. Determine the limits of the following.
   (a) $\left((3n)^{1/2n}\right)$, (b) $\left((1 + 1/2n)^{3n}\right)$.
9. Suppose that every subsequence of $X = (x_n)$ has a subsequence that converges to 0. Show that $\lim X = 0$.
10. Let (x_n) be a bounded sequence and for each $n \in \mathbb{N}$ let $s_n := \sup\{x_k : k \ge n\}$ and $S := \inf\{s_n\}$. Show that there exists a subsequence of (x_n) that converges to S.
11. Suppose that $x_n \ge 0$ for all $n \in \mathbb{N}$ and that $\lim((-1)^n x_n)$ exists. Show that (x_n) converges.
12. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$.
13. If $x_n := (-1)^n/n$, find the subsequence of (x_n) that is constructed in the second proof of the Bolzano-Weierstrass Theorem 3.4.8, when we take $I_1 := [-1, 1]$.

14. Let (x_n) be a bounded sequence and let $s := \sup\{x_n : n \in \mathbb{N}\}$. Show that if $s \notin \{x_n : n \in \mathbb{N}\}$, then there is a subsequence of (x_n) that converges to s.
15. Let (I_n) be a nested sequence of closed bounded intervals. For each $n \in \mathbb{N}$, let $x_n \in I_n$. Use the Bolzano-Weierstrass Theorem to give a proof of the Nested Intervals Property 2.5.2.
16. Give an example to show that Theorem 3.4.9 fails if the hypothesis that X is a bounded sequence is dropped.
17. Alternate the terms of the sequences $(1 + 1/n)$ and $(-1/n)$ to obtain the sequence (x_n) given by
$$(2, -1, 3/2, -1/2, 4/3, -1/3, 5/4, -1/4, \ldots).$$
Determine the values of $\lim\sup(x_n)$ and $\lim\inf(x_n)$. Also find $\sup\{x_n\}$ and $\inf\{x_n\}$.
18. Show that if (x_n) is a bounded sequence, then (x_n) converges if and only if $\lim\sup(x_n) = \lim\inf(x_n)$.
19. Show that if (x_n) and (y_n) are bounded sequences, then
$$\lim\sup(x_n + y_n) \le \lim\sup(x_n) + \lim\sup(y_n).$$
Give an example in which the two sides are not equal.

Section 3.5 The Cauchy Criterion

The Monotone Convergence Theorem is extraordinarily useful and important, but it has the significant drawback that it applies only to sequences that are monotone. It is important for us to have a condition implying the convergence of a sequence that does not require us to know the value of the limit in advance, and is not restricted to monotone sequences. The Cauchy Criterion, which will be established in this section, is such a condition.

3.5.1 Definition A sequence $X = (x_n)$ of real numbers is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists a natural number $H(\varepsilon)$ such that for all natural numbers $n, m \ge H(\varepsilon)$, the terms x_n, x_m satisfy $|x_n - x_m| < \varepsilon$.

The significance of the concept of Cauchy sequence lies in the main theorem of this section, which asserts that a sequence of real numbers is convergent if and only if it is a Cauchy sequence. This will give us a method of proving a sequence converges without knowing the limit of the sequence.

However, we will first highlight the definition of Cauchy sequence in the following examples.

3.5.2 Examples **(a)** The sequence $(1/n)$ is a Cauchy sequence.

If $\varepsilon > 0$ is given, we choose a natural number $H = H(\varepsilon)$ such that $H > 2/\varepsilon$. Then if $m, n \ge H$, we have $1/n \le 1/H < \varepsilon/2$ and similarly $1/m < \varepsilon/2$. Therefore, it follows that if m, $n \ge H$, then
$$\left|\frac{1}{n} - \frac{1}{m}\right| \le \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
Since $\varepsilon > 0$ is arbitrary, we conclude that $(1/n)$ is a Cauchy sequence.

(b) The sequence $(1 + (-1)^n)$ is *not* a Cauchy sequence.

The negation of the definition of Cauchy sequence is: There exists $\varepsilon_0 > 0$ such that for every H there exist at least one $n > H$ and at least one $m > H$ such that $|x_n - x_m| \ge \varepsilon_0$. For

To estimate the accuracy, we note that $|x_2 - x_1| < 0.2$. Thus, after n steps it follows from Corollary 3.5.10(i) that we are sure that $|x^* - x_n| \le 3^{n-1}(7^{n-2} \cdot 20)$. Thus, when $n = 6$, we are sure that

$$|x^* - x_6| \le 3^5/(7^4 \cdot 20) = 243/48\,020 < 0.0051.$$

Actually the approximation is substantially better than this. In fact, since $|x_6 - x_5| < 0.000\,0005$, it follows from 3.5.10(ii) that $|x^* - x_6| \le \frac{3}{4}|x_6 - x_5| < 0.000\,0004$. Hence the first five decimal places of x_6 are correct. □

Exercises for Section 3.5

1. Give an example of a bounded sequence that is not a Cauchy sequence.
2. Show directly from the definition that the following are Cauchy sequences.

   (a) $\left(\dfrac{n+1}{n}\right)$, (b) $\left(1 + \dfrac{1}{2!} + \cdots + \dfrac{1}{n!}\right)$.
3. Show directly from the definition that the following are not Cauchy sequences.

   (a) $((-1)^n)$, (b) $\left(n + \dfrac{(-1)^n}{n}\right)$, (c) $(\ln n)$
4. Show directly from the definition that if (x_n) and (y_n) are Cauchy sequences, then $(x_n + y_n)$ and $(x_n y_n)$ are Cauchy sequences.
5. If $x_n := \sqrt{n}$, show that (x_n) satisfies $\lim|x_{n+1} - x_n| = 0$, but that it is not a Cauchy sequence.
6. Let p be a given natural number. Give an example of a sequence (x_n) that is not a Cauchy sequence, but that satisfies $\lim|x_{n+p} - x_n| = 0$.
7. Let (x_n) be a Cauchy sequence such that x_n is an integer for every $n \in \mathbb{N}$. Show that (x_n) is ultimately constant.
8. Show directly that a bounded, monotone increasing sequence is a Cauchy sequence.
9. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence.
10. If $x_1 < x_2$ are arbitrary real numbers and $x_n := \frac{1}{2}(x_{n-2} + x_{n-1})$ for $n > 2$, show that (x_n) is convergent. What is its limit?
11. If $y_1 < y_2$ are arbitrary real numbers and $y_n := \frac{1}{3}y_{n-1} + \frac{2}{3}y_{n-2}$ for $n > 2$, show that (y_n) is convergent. What is its limit?
12. If $x_1 > 0$ and $x_{n+1} := (2 + x_n)^{-1}$ for $n \ge 1$, show that (x_n) is a contractive sequence. Find the limit.
13. If $x_1 := 2$ and $x_{n+1} := 2 + 1/x_n$ for $n \ge 1$, show that (x_n) is a contractive sequence. What is its limit?
14. The polynomial equation $x^3 - 5x + 1 = 0$ has a root r with $0 < r < 1$. Use an appropriate contractive sequence to calculate r within 10^{-4}.

Section 3.6 Properly Divergent Sequences

For certain purposes it is convenient to define what is meant for a sequence (x_n) of real numbers to "tend to $\pm\infty$."

Proof. (a) If $\lim(x_n) = +\infty$, and if $\alpha \in \mathbb{R}$ is given, then there exists a natural number $K(\alpha)$ such that if $n \geq K(\alpha)$, then $\alpha < x_n$. In view of (1), it follows that $\alpha < y_n$ for all $n \geq K(\alpha)$. Since α is arbitrary, it follows that $\lim(y_n) = +\infty$.

The proof of (b) is similar. Q.E.D.

Remarks **(a)** Theorem 3.6.4 remains true if condition (1) is ultimately true; that is, if there exists $m \in \mathbb{N}$ such that $x_n \leq y_n$ for all $n \geq m$.

(b) If condition (1) of Theorem 3.6.4 holds and if $\lim(y_n) = +\infty$, it does *not* follow that $\lim(x_n) = +\infty$. Similarly, if (1) holds and if $\lim(x_n) = -\infty$, it does *not* follow that $\lim(y_n) = -\infty$. In using Theorem 3.6.4 to show that a sequence tends to $+\infty$ [respectively, $-\infty$] we need to show that the terms of the sequence are ultimately greater [respectively, less] than or equal to the corresponding terms of a sequence that is known to tend to $+\infty$ [respectively, $-\infty$].

Since it is sometimes difficult to establish an inequality such as (1), the following "limit comparison theorem" is often more convenient to use than Theorem 3.6.4.

3.6.5 Theorem *Let (x_n) and (y_n) be two sequences of positive real numbers and suppose that for some $L \in \mathbb{R}, L > 0$, we have*

$$\lim(x_n/y_n) = L. \tag{2}$$

Then $\lim(x_n) = +\infty$ *if and only if* $\lim(y_n) = +\infty$.

Proof. If (2) holds, there exists $K \in \mathbb{N}$ such that

$$\tfrac{1}{2}L < x_n/y_n < \tfrac{3}{2}L \quad \text{for all} \quad n \geq K.$$

Hence we have $(\frac{1}{2}L)y_n < x_n < (\frac{3}{2}L)y_n$ for all $n \geq K$. The conclusion now follows from a slight modification of Theorem 3.6.4. We leave the details to the reader. Q.E.D.

The reader can show that the conclusion need not hold if either $L = 0$ or $L = +\infty$. However, there are some partial results that can be established in these cases, as will be seen in the exercises.

Exercises for Section 3.6

1. Show that if (x_n) is an unbounded sequence, then there exists a properly divergent subsequence.
2. Give examples of properly divergent sequences (x_n) and (y_n) with $y_n \neq 0$ for all $n \in \mathbb{N}$ such that:
   (a) (x_n/y_n) is convergent, (b) (x_n/y_n) is properly divergent.
3. Show that if $x_n > 0$ for all $n \in \mathbb{N}$, then $\lim(x_n) = 0$ if and only if $\lim(1/x_n) = +\infty$.
4. Establish the proper divergence of the following sequences.
   (a) $(\sqrt{n})$, (b) $(\sqrt{n+1})$,
   (c) $(\sqrt{n-1})$, (d) $(n/\sqrt{n+1})$.
5. Is the sequence $(n \sin n)$ properly divergent?
6. Let (x_n) be properly divergent and let (y_n) be such that $\lim(x_n y_n)$ belongs to $\mathbb{R}$. Show that (y_n) converges to 0.
7. Let (x_n) and (y_n) be sequences of positive numbers such that $\lim(x_n/y_n) = 0$.
   (a) Show that if $\lim(x_n) = +\infty$, then $\lim(y_n) = +\infty$.
   (b) Show that if (y_n) is bounded, then $\lim(x_n) = 0$.

8. Investigate the convergence or the divergence of the following sequences:
   (a) $(\sqrt{n^2+2})$, (b) $(\sqrt{n}/(n^2+1))$,
   (c) $(\sqrt{n^2+1}/\sqrt{n})$, (d) $(\sin\sqrt{n})$.
9. Let (x_n) and (y_n) be sequences of positive numbers such that $\lim(x_n/y_n) = +\infty$,
   (a) Show that if $\lim(y_n) = +\infty$, then $\lim(x_n) = +\infty$.
   (b) Show that if (x_n) is bounded, then $\lim(y_n) = 0$.
10. Show that if $\lim(a_n/n) = L$, where $L > 0$, then $\lim(a_n) = +\infty$.

Section 3.7 Introduction to Infinite Series

We will now give a brief introduction to infinite series of real numbers. This is a topic that will be discussed in more detail in Chapter 9, but because of its importance, we will establish a few results here. These results will be seen to be immediate consequences of theorems we have met in this chapter.

In elementary texts, an infinite series is sometimes "defined" to be "an expression of the form"

(1) $$x_1 + x_2 + \cdots + x_n + \cdots.$$

However, this "definition" lacks clarity, since there is *a priori* no particular value that we can attach to this array of symbols, which calls for an *infinite* number of additions to be performed.

3.7.1 Definition If $X := (x_n)$ is a sequence in $\mathbb{R}$, then the **infinite series** (or simply the **series**) **generated by** X is the sequence $S := (s_k)$ defined by

$$\begin{aligned} s_1 &:= x_1 \\ s_2 &:= s_1 + x_2 \quad (= x_1 + x_2) \\ &\cdots \\ s_k &:= s_{k-1} + x_k \quad (= x_1 + x_2 + \cdots + x_k) \\ &\cdots \end{aligned}$$

The numbers x_n are called the **terms** of the series and the numbers s_k are called the **partial sums** of this series. If $\lim S$ exists, we say that this series is **convergent** and call this limit the **sum** or the **value** of this series. If this limit does not exist, we say that the series S is **divergent.**

It is convenient to use symbols such as

(2) $$\sum(x_n) \quad \text{or} \quad \sum x_n \quad \text{or} \quad \sum_{n=1}^{\infty} x_n$$

to denote both the infinite series S generated by the sequence $X = (x_n)$ and also to denote the value $\lim S$, in case this limit exists. Thus the symbols in (2) may be regarded merely as a way of exhibiting an infinite series whose convergence or divergence is to be investigated. In practice, this double use of these notations does not lead to any confusion, provided it is understood that the convergence (or divergence) of the series must be established.

were true, we could argue as in (a). However, (10) is *false* for all $n \in \mathbb{N}$. The reader can probably show that the inequality

$$0 < \frac{1}{n^2 - n + 1} \le \frac{2}{n^2}$$

is valid for all $n \in \mathbb{N}$, and this inequality will work just as well. However, it might take some experimentation to think of such an inequality and then establish it.

Instead, if we take $x_n := 1/(n^2 - n + 1)$ and $y_n := 1/n^2$, then we have

$$\frac{x_n}{y_n} = \frac{n^2}{n^2 - n + 1} = \frac{1}{1 - (1/n) + (1/n^2)} \to 1.$$

Therefore, the convergence of the given series follows from the Limit Comparison Test 3.7.8(a).

(c) The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ is divergent.

This series closely resembles the series $\sum 1/\sqrt{n}$, which is a p-series with $p = \frac{1}{2}$; by Example 3.7.6(e), it is divergent. If we let $x_n := 1/\sqrt{n+1}$ and $y_n := 1/\sqrt{n}$, then we have

$$\frac{x_n}{y_n} = \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{1}{\sqrt{1 + 1/n}} \to 1.$$

Therefore the Limit Comparison Test 3.7.8(a) applies.

(d) The series $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent.

It would be possible to establish this convergence by showing (by Induction) that $n^2 < n!$ for $n \ge 4$, whence it follows that

$$0 < \frac{1}{n!} < \frac{1}{n^2} \quad \text{for} \quad n \ge 4.$$

Alternatively, if we let $x := 1/n!$ and $y_n := 1/n^2$, then (when $n \ge 4$) we have

$$0 \le \frac{x_n}{y_n} = \frac{n^2}{n!} = \frac{n}{1 \cdot 2 \cdots (n-1)} < \frac{1}{n-2} \to 0.$$

Therefore the Limit Comparison Test 3.7.8(b) applies. (Note that this test was a bit troublesome to apply since we do not presently know the convergence of any series for which the limit of x_n/y_n is really easy to determine.) □

Exercises for Section 3.7

1. Let $\sum a_n$ be a given series and let $\sum b_n$ be the series in which the terms are the same and in the same order as in $\sum a_n$ except that the terms for which $a_n = 0$ have been omitted. Show that $\sum a_n$ converges to A if and only if $\sum b_n$ converges to A.
2. Show that the convergence of a series is not affected by changing a *finite* number of its terms. (Of course, the value of the sum may be changed.)
3. By using partial fractions, show that

   (a) $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1$, (b) $\sum_{n=0}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha} > 0$, if $\alpha > 0$.

   (c) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$.
4. If $\sum x_n$ and $\sum y_n$ are convergent, show that $\sum (x_n + y_n)$ is convergent.

5. Can you give an example of a convergent series $\sum x_n$ and a divergent series $\sum y_n$ such that $\sum (x_n + y_n)$ is convergent? Explain.

6. (a) Calculate the value of $\sum_{n=2}^{\infty} (2/7)^n$. (Note the series starts at $n = 2$.)

   (b) Calculate the value of $\sum_{n=1}^{\infty} (1/3)^{2n}$. (Note the series starts at $n = 1$.)

7. Find a formula for the series $\sum_{n=1}^{\infty} r^{2n}$ when $|r| < 1$.

8. Let $r_1, r_2, \ldots, r_n, \ldots$ be an enumeration of the rational numbers in the interval $[0,1]$. (See Section 1.3.) For a given $\varepsilon > 0$, put an interval of length ε^n about the nth rational number r_n for $n = 1, 2, 3, \ldots$, and find the total sum of the lengths of all the intervals. Evaluate this number for $\varepsilon = 0.1$ and $\varepsilon = 0.01$.

9. (a) Show that the series $\sum_{n=1}^{\infty} \cos n$ is divergent.

   (b) Show that the series $\sum_{n=1}^{\infty} (\cos n)/n^2$ is convergent.

10. Use an argument similar to that in Example 3.7.6(f) to show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent.

11. If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum a_n^2$ always convergent? Either prove it or give a counterexample.

12. If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum \sqrt{a_n}$ always convergent? Either prove it or give a counterexample.

13. If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum \sqrt{a_n a_{n+1}}$ always convergent? Either prove it or give a counterexample.

14. If $\sum a_n$ with $a_n > 0$ is convergent, and if $b_n := (a_1 + \cdots + a_n)/n$ for $n \in \mathbb{N}$, then show that $\sum b_n$ is always divergent.

15. Let $\sum_{n=1}^{\infty} a(n)$ be such that $(a(n))$ is a decreasing sequence of strictly positive numbers. If $s(n)$ denotes the nth partial sum, show (by grouping the terms in $s(2^n)$ in two different ways) that

    $$\tfrac{1}{2}(a(1) + 2a(2) + \cdots + 2^n a(2^n)) \le s(2^n) \le \big(a(1) + 2a(2) + \cdots + 2^{n-1}a(2^{n-1})\big) + a(2^n).$$

    Use these inequalities to show that $\sum_{n=1}^{\infty} a(n)$ converges if and only if $\sum_{n=1}^{\infty} 2^n a(2^n)$ converges. This result is often called the **Cauchy Condensation Test**; it is very powerful.

16. Use the Cauchy Condensation Test to discuss the p-series $\sum_{n=1}^{\infty} (1/n^p)$ for $p > 0$.

17. Use the Cauchy Condensation Test to establish the divergence of the series:

    (a) $\sum \dfrac{1}{n \ln n}$, (b) $\sum \dfrac{1}{n(\ln n)(\ln \ln n)}$,

    (c) $\sum \dfrac{1}{n(\ln n)(\ln \ln n)(\ln \ln \ln n)}$.

18. Show that if $c > 1$, then the following series are convergent:

    (a) $\sum \dfrac{1}{n(\ln n)^c}$, (b) $\sum \dfrac{1}{n(\ln n)(\ln \ln n)^c}$.

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Exercises for Section 4.1

1. Determine a condition on $|x-1|$ that will assure that:
   (a) $|x^2-1| < \frac{1}{2}$, (b) $|x^2-1| < 1/10^{-3}$,
   (c) $|x^2-1| < 1/n$ for a given $n \in \mathbb{N}$, (d) $|x^3-1| < 1/n$ for a given $n \in \mathbb{N}$.
2. Determine a condition on $|x-4|$ that will assure that:
   (a) $|\sqrt{x}-2| < \frac{1}{2}$, (b) $|\sqrt{x}-2| < 10^{-2}$.
3. Let c be a cluster point of $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Prove that $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c} |f(x) - L| = 0$.
4. Let $f := \mathbb{R} \to \mathbb{R}$ and let $c \in \mathbb{R}$. Show that $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to 0} f(x+c) = L$.
5. Let $I := (0, a)$ where $a > 0$, and let $g(x) := x^2$ for $x \in I$. For any points $x, c \in I$, show that $|g(x) - c^2| \le 2a|x-c|$. Use this inequality to prove that $\lim_{x \to c} x^2 = c^2$ for any $c \in I$.
6. Let I be an interval in $\mathbb{R}$, let $f : I \to \mathbb{R}$, and let $c \in I$. Suppose there exist constants K and L such that $|f(x) - L| \le K|x-c|$ for $x \in I$. Show that $\lim_{x \to c} f(x) = L$.
7. Show that $\lim_{x \to c} x^3 = c^3$ for any $c \in \mathbb{R}$.
8. Show that $\lim_{x \to c} \sqrt{x} = \sqrt{c}$ for any $c > 0$.
9. Use either the ε-δ definition of limit or the Sequential Criterion for limits, to establish the following limits.
   (a) $\lim_{x \to 2} \dfrac{1}{1-x} = -1$, (b) $\lim_{x \to 1} \dfrac{x}{1+x} = \dfrac{1}{2}$,
   (c) $\lim_{x \to 0} \dfrac{x^2}{|x|} = 0$, (d) $\lim_{x \to 1} \dfrac{x^2 - x + 1}{x+1} = \dfrac{1}{2}$.
10. Use the definition of limit to show that
    (a) $\lim_{x \to 2} (x^2 + 4x) = 12$, (b) $\lim_{x \to -1} \dfrac{x+5}{2x+3} = 4$.
11. Use the definition of limit to prove the following.
    (a) $\lim_{x \to 3} \dfrac{2x+3}{4x-9} = 3$, (b) $\lim_{x \to 6} \dfrac{x^2 - 3x}{x+3} = 2$.
12. Show that the following limits do *not* exist.
    (a) $\lim_{x \to 0} \dfrac{1}{x^2}$ $(x > 0)$, (b) $\lim_{x \to 0} \dfrac{1}{\sqrt{x}}$ $(x > 0)$,
    (c) $\lim_{x \to 0} (x + \operatorname{sgn}(x))$, (d) $\lim_{x \to 0} \sin(1/x^2)$.
13. Suppose the function $f : \mathbb{R} \to \mathbb{R}$ has limit L at 0, and let $a > 0$. If $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) := f(ax)$ for $x \in \mathbb{R}$, show that $\lim_{x \to 0} g(x) = L$.
14. Let $c \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x \to c} (f(x))^2 = L$.
    (a) Show that if $L = 0$, then $\lim_{x \to c} f(x) = 0$.
    (b) Show by example that if $L \ne 0$, then f may not have a limit at c.
15. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) := x$ if x is rational, and $f(x) = 0$ if x is irrational.
    (a) Show that f has a limit at $x = 0$.
    (b) Use a sequential argument to show that if $c \ne 0$, then f does not have a limit at c.
16. Let $f : \mathbb{R} \to \mathbb{R}$, let I be an *open* interval in $\mathbb{R}$, and let $c \in I$. If f_1 is the restriction of f to I, show that f_1 has a limit at c if and only if f has a limit at c, and that the limits are equal.
17. Let $f : \mathbb{R} \to \mathbb{R}$, let J be a *closed* interval in $\mathbb{R}$, and let $c \in J$. If f_2 is the restriction of f to J, show that if f has a limit at c then f_2 has a limit at c. Show by example that it does *not* follow that if f_2 has a limit at c, then f has a limit at c.

Exercises for Section 4.2

1. Apply Theorem 4.2.4 to determine the following limits:
   (a) $\lim_{x\to 1}(x+1)(2x+3) \quad (x\in\mathbb{R})$, (b) $\lim_{x\to 1}\dfrac{x^2+2}{x^2-2} \quad (x>0)$,
   (c) $\lim_{x\to 2}\left(\dfrac{1}{x+1}-\dfrac{1}{2x}\right) \quad (x>0)$, (d) $\lim_{x\to 0}\dfrac{x+1}{x^2+2} \quad (x\in\mathbb{R})$.
2. Determine the following limits and state which theorems are used in each case. (You may wish to use Exercise 15 below.)
   (a) $\lim_{x\to 2}\sqrt{\dfrac{2x+1}{x+3}} \quad (x>0)$, (b) $\lim_{x\to 2}\dfrac{x^2-4}{x-2} \quad (x>0)$,
   (c) $\lim_{x\to 0}\dfrac{(x+1)^2-1}{x} \quad (x>0)$, (d) $\lim_{x\to 1}\dfrac{\sqrt{x}-1}{x-1} \quad (x>0)$.
3. Find $\lim_{x\to 0}\dfrac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2}$ where $x>0$.
4. Prove that $\lim_{x\to 0}\cos(1/x)$ does not exist but that $\lim_{x\to 0}x\cos(1/x)=0$.
5. Let f, g be defined on $A\subseteq\mathbb{R}$ to $\mathbb{R}$, and let c be a cluster point of A. Suppose that f is bounded on a neighborhood of c and that $\lim_{x\to c}g=0$. Prove that $\lim_{x\to c}fg=0$.
6. Use the definition of the limit to prove the first assertion in Theorem 4.2.4(a).
7. Use the sequential formulation of the limit to prove Theorem 4.2.4(b).
8. Let $n\in\mathbb{N}$ be such that $n\geq 3$. Derive the inequality $-x^2\leq x^n\leq x^2$ for $-1<x<1$. Then use the fact that $\lim_{x\to 0}x^2=0$ to show that $\lim_{x\to 0}x^n=0$.
9. Let f, g be defined on A to $\mathbb{R}$ and let c be a cluster point of A.
   (a) Show that if both $\lim_{x\to c}f$ and $\lim_{x\to c}(f+g)$ exist, then $\lim_{x\to c}g$ exists.
   (b) If $\lim_{x\to c}f$ and $\lim_{x\to c}fg$ exist, does it follow that $\lim_{x\to c}g$ exists?
10. Give examples of functions f and g such that f and g do not have limits at a point c, but such that both $f+g$ and fg have limits at c.
11. Determine whether the following limits exist in $\mathbb{R}$.
    (a) $\lim_{x\to 0}\sin(1/x^2) \quad (x\neq 0)$, (b) $\lim_{x\to 0}x\sin(1/x^2) \quad (x\neq 0)$,
    (c) $\lim_{x\to 0}\operatorname{sgn}\sin(1/x) \quad (x\neq 0)$, (d) $\lim_{x\to 0}\sqrt{x}\sin(1/x^2) \quad (x>0)$.
12. Let $f:\mathbb{R}\to\mathbb{R}$ be such that $f(x+y)=f(x)+f(y)$ for all x, y in $\mathbb{R}$. Assume that $\lim_{x\to 0}f=L$ exists. Prove that $L=0$, and then prove that f has a limit at every point $c\in\mathbb{R}$. [*Hint*: First note that $f(2x)=f(x)+f(x)=2f(x)$ for $x\in\mathbb{R}$. Also note that $f(x)=f(x-c)+f(c)$ for x, c in $\mathbb{R}$.]
13. Functions f and g are defined on R by $f(x):=x+1$ and $g(x):=2$ if $x\neq 1$ and $g(1):=0$.
    (a) Find $\lim_{x\to 1}g(f(x))$ and compare with the value of $g(\lim_{x\to 1}f(x))$.
    (b) Find $\lim_{x\to 1}f(g(x))$ and compare with the value of $f(\lim_{x\to 1}g(x))$.
14. Let $A\subseteq\mathbb{R}$, let $f:A\to\mathbb{R}$ and let $c\in\mathbb{R}$ be a cluster point of A. If $\lim_{x\to c}f$ exists, and if $|f|$ denotes the function defined for $x\in A$ by $|f|(x):=|f(x)|$, prove that $\lim_{x\to c}|f|=\left|\lim_{x\to c}f\right|$.
15. Let $A\subseteq\mathbb{R}$, let $f:A\to\mathbb{R}$, and let $c\in\mathbb{R}$ be a cluster point of A. In addition, suppose that $f(x)\geq 0$ for all $x\in A$, and let $\sqrt{f}$ be the function defined for $x\in A$ by $(\sqrt{f})(x):=\sqrt{f(x)}$. If $\lim_{x\to c}f$ exists, prove that $\lim_{x\to c}\sqrt{f}=\sqrt{\lim_{x\to c}f}$.

Section 4.3 Some Extensions of the Limit Concept†

In this section, we shall present three types of extensions of the notion of a limit of a function that often occur. Since all the ideas here are closely parallel to ones we have already encountered, this section can be read easily.

† This section can be largely omitted on a first reading of this chapter.

Exercises for Section 4.3

1. Prove Theorem 4.3.2.
2. Give an example of a function that has a right-hand limit but not a left-hand limit at a point.
3. Let $f(x) := |x|^{-1/2}$ for $x \neq 0$. Show that $\lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0-} f(x) = +\infty$.
4. Let $c \in \mathbb{R}$ and let f be defined for $x \in (c, \infty)$ and $f(x) > 0$ for all $x \in (c, \infty)$. Show that $\lim\limits_{x\to c} f = \infty$ if and only if $\lim\limits_{x\to c} 1/f = 0$.
5. Evaluate the following limits, or show that they do not exist.

   (a) $\lim\limits_{x\to 1+} \dfrac{x}{x-1} \quad (x \neq 1)$,

   (b) $\lim\limits_{x\to 1} \dfrac{x}{x-1} \quad (x \neq 1)$,

   (c) $\lim\limits_{x\to 0+} (x+2)/\sqrt{x} \quad (x > 0)$,

   (d) $\lim\limits_{x\to \infty} (x+2)/\sqrt{x} \quad (x > 0)$,

   (e) $\lim\limits_{x\to 0} (\sqrt{x+1})/x \quad (x > -1)$,

   (f) $\lim\limits_{x\to \infty} (\sqrt{x+1})/x \quad (x > 0)$,

   (g) $\lim\limits_{x\to \infty} \dfrac{\sqrt{x}-5}{\sqrt{x}+3} \quad (x > 0)$,

   (h) $\lim\limits_{x\to \infty} \dfrac{\sqrt{x}-x}{\sqrt{x}+x} \quad (x > 0)$.
6. Prove Theorem 4.3.11.
7. Suppose that f and g have limits in $\mathbb{R}$ as $x \to \infty$ and that $f(x) \leq g(x)$ for all $x \in (a, \infty)$. Prove that $\lim\limits_{x\to\infty} f \leq \lim\limits_{x\to\infty} g$.
8. Let f be defined on $(0, \infty)$ to $\mathbb{R}$. Prove that $\lim\limits_{x\to\infty} f(x) = L$ if and only if $\lim\limits_{x\to 0+} f(1/x) = L$.
9. Show that if $f : (a, \infty) \to \mathbb{R}$ is such that $\lim\limits_{x\to\infty} xf(x) = L$ where $L \in \mathbb{R}$, then $\lim\limits_{x\to\infty} f(x) = 0$.
10. Prove Theorem 4.3.14.
11. Suppose that $\lim\limits_{x\to c} f(x) = L$ where $L > 0$, and that $\lim\limits_{x\to c} g(x) = \infty$. Show that $\lim\limits_{x\to c} f(x)g(x) = \infty$. If $L = 0$, show by example that this conclusion may fail.
12. Find functions f and g defined on $(0, \infty)$ such that $\lim\limits_{x\to\infty} f = \infty$ and $\lim\limits_{x\to\infty} g = \infty$, and $\lim\limits_{x\to\infty} (f - g) = 0$. Can you find such functions, with $g(x) > 0$ for all $x \in (0, \infty)$, such that $\lim\limits_{x\to\infty} f/g = 0$?
13. Let f and g be defined on (a, ∞) and suppose $\lim\limits_{x\to\infty} f = L$ and $\lim\limits_{x\to\infty} g = \infty$. Prove that $\lim\limits_{x\to\infty} f \circ g = L$.

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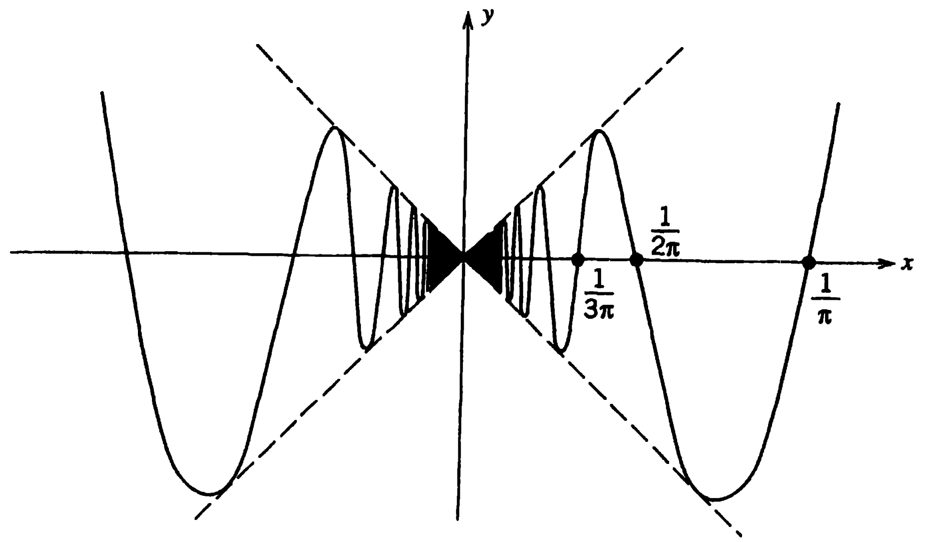

Figure 5.1.3 Graph of $f(x) = x\ \sin(1/x)$ $(x \neq 0)$

Exercises for Section 5.1

1. Prove the Sequential Criterion 5.1.3.
2. Establish the Discontinuity Criterion 5.1.4.
3. Let $a < b < c$. Suppose that f is continuous on $[a, b]$, that g is continuous on $[b, c]$, and that $f(b) = g(b)$. Define h on $[a, c]$ by $h(x) := f(x)$ for $x \in [a, b]$ and $h(x) := g(x)$ for $x \in [b, c]$. Prove that h is continuous on $[a, c]$.
4. If $x \in \mathbb{R}$, we define $[\![x]\!]$ to be the greatest integer $n \in \mathbb{Z}$ such that $n \leq x$. (Thus, for example, $[\![8.3]\!] = 8, [\![\pi]\!] = 3, [\![-\pi]\!] = -4$.) The function $x \mapsto [\![x]\!]$ is called the **greatest integer function.** Determine the points of continuity of the following functions:
   (a) $f(x) := [\![x]\!]$, (b) $g(x) := x[\![x]\!]$,
   (c) $h(x) := [\![\sin x]\!]$, (d) $k(x) := [\![1/x]\!]$ $(x \neq 0)$.
5. Let f be defined for all $x \in \mathbb{R}, x \neq 2$, by $f(x) = (x^2 + x - 6)/(x - 2)$. Can f be defined at $x = 2$ in such a way that f is continuous at this point?
6. Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$ be continuous at a point $c \in A$. Show that for any $\varepsilon > 0$, there exists a neighborhood $V_\delta(c)$ of c such that if $x, y \in A \cap V_\delta(c)$, then $|f(x) - f(y)| < \varepsilon$.
7. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at c and let $f(c) > 0$. Show that there exists a neighborhood $V_\delta(c)$ of c such that if $x \in V_\delta(c)$, then $f(x) > 0$.
8. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $S := \{x \in \mathbb{R} : f(x) = 0\}$ be the "zero set" of f. If (x_n) is in S and $x = \lim(x_n)$, show that $x \in S$.
9. Let $A \subseteq B \subseteq \mathbb{R}$, let $f : B \to \mathbb{R}$ and let g be the restriction of f to A (that is, $g(x) = f(x)$ for $x \in A$).
   (a) If f is continuous at $c \in A$, show that g is continuous at c.
   (b) Show by example that if g is continuous at c, it need not follow that f is continuous at c.
10. Show that the absolute value function $f(x) := |x|$ is continuous at every point $c \in \mathbb{R}$.
11. Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$. Show that f is continuous at every point $c \in \mathbb{R}$.
12. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number r. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.
13. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := 2x$ for x rational, and $g(x) := x + 3$ for x irrational. Find all points at which g is continuous.

14. Let $A := (0, \infty)$ and let $k : A \to \mathbb{R}$ be defined as follows. For $x \in A, x$ irrational, we define $k(x) = 0$; for $x \in A$ rational and of the form $x = m/n$ with natural numbers m, n having no common factors except 1, we define $k(x) := n$. Prove that k is unbounded on every open interval in A. Conclude that k is not continuous at any point of A. (See Example 5.1.6(h).)
15. Let $f : (0, 1) \to \mathbb{R}$ be bounded but such that $\lim_{x \to 0} f$ does not exist. Show that there are two sequences (x_n) and (y_n) in (0, 1) with $\lim(x_n) = 0 = \lim(y_n)$, but such that $(f(x_n))$ and $(f(y_n))$ exist but are not equal.

Section 5.2 Combinations of Continuous Functions

Let $A \subseteq \mathbb{R}$ and let f and g be functions that are defined on A to $\mathbb{R}$ and let $b \in \mathbb{R}$. In Definition 4.2.3 we defined the sum, difference, product, and multiple functions denoted by $f + g, f - g, fg, bf$. In addition, if $h : A \to \mathbb{R}$ is such that $h(x) \neq 0$ for all $x \in A$, then we defined the quotient function denoted by f/h.

The next result is similar to Theorem 4.2.4, from which it follows.

5.2.1 Theorem *Let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let $b \in \mathbb{R}$. Suppose that $c \in A$ and that f and g are continuous at c.*

(a) *Then $f + g, f - g, fg$, and bf are continuous at c.*

(b) *If $h : A \to \mathbb{R}$ is continuous at $c \in A$ and if $h(x) \neq 0$ for all $x \in A$, then the quotient f/h is continuous at c.*

Proof. If $c \in A$ is not a cluster point of A, then the conclusion is automatic. Hence we assume that c is a cluster point of A.

(a) Since f and g are continuous at c, then

$$f(c) = \lim_{x \to c} f \quad \text{and} \quad g(c) = \lim_{x \to c} g.$$

Hence it follows from Theorem 4.2.4(a) that

$$(f + g)(c) = f(c) + g(c) = \lim_{x \to c} (f + g).$$

Therefore $f + g$ is continuous at c. The remaining assertions in part (a) are proved in a similar fashion.

(b) Since $c \in A$, then $h(c) \neq 0$. But since $h(c) = \lim_{x \to c} h$, it follows from Theorem 4.2.4(b) that

$$\frac{f}{h}(c) = \frac{f(c)}{h(c)} = \frac{\lim_{x \to c} f}{\lim_{x \to c} h} = \lim_{x \to c} \left(\frac{f}{h}\right).$$

Therefore f/h is continuous at c. Q.E.D.

The next result is an immediate consequence of Theorem 5.2.1, applied to every point of A. However, since it is an extremely important result, we shall state it formally.

5.2.2 Theorem *Let $A \subseteq \mathbb{R}$, let f and g be continuous on A to $\mathbb{R}$, and let $b \in \mathbb{R}$.*

(a) *The functions $f + g, f - g, fg$, and bf are continuous on A.*

(b) *If $h : A \to \mathbb{R}$ is continuous on A and $h(x) \neq 0$ for $x \in A$, then the quotient f/h is continuous on A.*

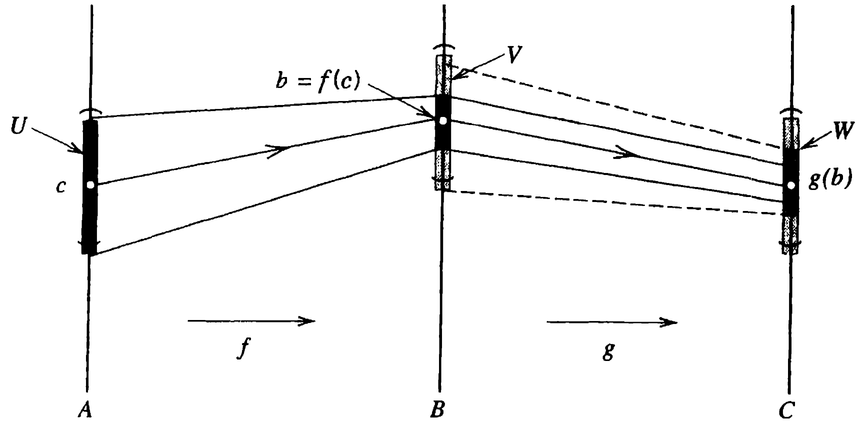

Figure 5.2.1 The composition of f and g

Theorems 5.2.6 and 5.2.7 are very useful in establishing that certain functions are continuous. They can be used in many situations where it would be difficult to apply the definition of continuity directly.

5.2.8 Examples **(a)** Let $g_1(x) := |x|$ for $x \in \mathbb{R}$. It follows from the Triangle Inequality that

$$|g_1(x) - g_1(c)| \leq |x - c|$$

for all $x, c \in \mathbb{R}$. Hence g_1 is continuous at $c \in \mathbb{R}$. If $f : A \to \mathbb{R}$ is any function that is continuous on A, then Theorem 5.2.7 implies that $g_1 \circ f = |f|$ is continuous on A. This gives another proof of Theorem 5.2.4.

(b) Let $g_2(x) := \sqrt{x}$ for $x \geq 0$. It follows from Theorems 3.2.10 and 5.1.3 that g_2 is continuous at any number $c \geq 0$. If $f : A \to \mathbb{R}$ is continuous on A and if $f(x) \geq 0$ for all $x \in A$, then it follows from Theorem 5.2.7 that $g_2 \circ f = \sqrt{f}$ is continuous on A. This gives another proof of Theorem 5.2.5.

(c) Let $g_3(x) := \sin x$ for $x \in \mathbb{R}$. We have seen in Example 5.2.3(c) that g_3 is continuous on $\mathbb{R}$. If $f : A \to \mathbb{R}$ is continuous on A, then it follows from Theorem 5.2.7 that $g_3 \circ f$ is continuous on A.

In particular, if $f(x) := 1/x$ for $x \neq 0$, then the function $g(x) := \sin(1/x)$ is continuous at every point $c \neq 0$. [We have seen, in Example 5.1.8(a), that g cannot be defined at 0 in order to become continuous at that point.] □

Exercises for Section 5.2

1. Determine the points of continuity of the following functions and state which theorems are used in each case.

   (a) $f(x) := \dfrac{x^2 + 2x + 1}{x^2 + 1} \quad (x \in \mathbb{R})$, (b) $g(x) := \sqrt{x + \sqrt{x}} \quad (x \geq 0)$,

   (c) $h(x) := \dfrac{\sqrt{1 + |\sin x|}}{x} \quad (x \neq 0)$, (d) $k(x) := \cos\sqrt{1 + x^2} \quad (x \in \mathbb{R})$.

2. Show that if $f : A \to \mathbb{R}$ is continuous on $A \subseteq \mathbb{R}$ and if $n \in \mathbb{N}$, then the function f^n defined by $f^n(x) = (f(x))^n$, for $x \in A$, is continuous on A.

3. Give an example of functions f and g that are both discontinuous at a point c in $\mathbb{R}$ such that (a) the sum $f + g$ is continuous at c, (b) the product fg is continuous at c.

4. Let $x \mapsto [\![x]\!]$ denote the greatest integer function (see Exercise 5.1.4). Determine the points of continuity of the function $f(x) := x - [\![x]\!], x \in \mathbb{R}$.
5. Let g be defined on $\mathbb{R}$ by $g(1) := 0$, and $g(x) := 2$ if $x \neq 1$, and let $f(x) := x + 1$ for all $x \in \mathbb{R}$. Show that $\lim_{x \to 0} g \circ f \neq (g \circ f)(0)$. Why doesn't this contradict Theorem 5.2.6?
6. Let f, g be defined on $\mathbb{R}$ and let $c \in \mathbb{R}$. Suppose that $\lim_{x \to c} f = b$ and that g is continuous at b. Show that $\lim_{x \to c} g \circ f = g(b)$. (Compare this result with Theorem 5.2.7 and the preceding exercise.)
7. Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is discontinuous at every point of $[0, 1]$ but such that $|f|$ is continuous on $[0, 1]$.
8. Let f, g be continuous from $\mathbb{R}$ to $\mathbb{R}$, and suppose that $f(r) = g(r)$ for all rational numbers r. Is it true that $f(x) = g(x)$ for all $x \in \mathbb{R}$?
9. Let $h : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ satisfying $h(m/2^n) = 0$ for all $m \in \mathbb{Z}, n \in \mathbb{N}$. Show that $h(x) = 0$ for all $x \in \mathbb{R}$.
10. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$, and let $P := \{x \in \mathbb{R} : f(x) > 0\}$. If $c \in P$, show that there exists a neighborhood $V_\delta(c) \subseteq P$.
11. If f and g are continuous on $\mathbb{R}$, let $S := \{x \in \mathbb{R} : f(x) \geq g(x)\}$. If $(s_n) \subseteq S$ and $\lim(s_n) = s$, show that $s \in S$.
12. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **additive** if $f(x + y) = f(x) + f(y)$ for all x, y in $\mathbb{R}$. Prove that if f is continuous at some point x_0, then it is continuous at every point of $\mathbb{R}$. (See Exercise 4.2.12.)
13. Suppose that f is a continuous additive function on $\mathbb{R}$. If $c := f(1)$, show that we have $f(x) = cx$ for all $x \in \mathbb{R}$. [*Hint*: First show that if r is a rational number, then $f(r) = cr$.]
14. Let $g : \mathbb{R} \to \mathbb{R}$ satisfy the relation $g(x + y) = g(x)g(y)$ for all x, y in $\mathbb{R}$. Show that if g is continuous at $x = 0$, then g is continuous at every point of $\mathbb{R}$. Also if we have $g(a) = 0$ for some $a \in \mathbb{R}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.
15. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous at a point c, and let $h(x) := \sup\{f(x), g(x)\}$ for $x \in \mathbb{R}$. Show that $h(x) = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|$ for all $x \in \mathbb{R}$. Use this to show that h is continuous at c.

Section 5.3 Continuous Functions on Intervals

Functions that are continuous on intervals have a number of very important properties that are not possessed by general continuous functions. In this section, we will establish some deep results that are of considerable importance and that will be applied later. Alternative proofs of these results will be given in Section 5.5.

5.3.1 Definition A function $f : A \to \mathbb{R}$ is said to be **bounded on** A if there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$.

In other words, a function is bounded on a set if its range is a bounded set in $\mathbb{R}$. To say that a function is *not* bounded on a given set is to say that no particular number can serve as a bound for its range. In exact language, a function f is not bounded on the set A if given any $M > 0$, there exists a point $x_M \in A$ such that $|f(x_M)| > M$. We often say that f is **unbounded on** A in this case.

For example, the function f defined on the interval $A := (0, \infty)$ by $f(x) := 1/x$ is not bounded on A because for any $M > 0$ we can take the point $x_M := 1/(M + 1)$ in A to get $f(x_M) = 1/x_M = M + 1 > M$. This example shows that continuous functions need not be

To prove the Preservation of Intervals Theorem 5.3.10, we will use Theorem 2.5.1 characterizing intervals.

5.3.10 Preservation of Intervals Theorem *Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I)$ is an interval.*

Proof. Let $\alpha, \beta \in f(I)$ with $a < \beta$; then there exist points $a, b \in I$ such that $\alpha = f(a)$ and $\beta = f(b)$. Further, it follows from Bolzano's Intermediate Value Theorem 5.3.7 that if $k \in (\alpha, \beta)$ then there exists a number $c \in I$ with $k = f(c) \in f(I)$. Therefore $[\alpha, \beta] \subseteq f(I)$, showing that $f(I)$ possesses property (1) of Theorem 2.5.1. Therefore $f(I)$ is an interval. Q.E.D.

Exercises for Section 5.3

1. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for each x in I. Prove that there exists a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in I$.
2. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be continuous functions on I. Show that the set $E := \{x \in I : f(x) = g(x)\}$ has the property that if $(x_n) \subseteq E$ and $x_n \to x_0$, then $x_0 \in E$.
3. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function on I such that for each x in I there exists y in I such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Prove there exists a point c in I such that $f(c) = 0$.
4. Show that every polynomial of odd degree with real coefficients has at least one real root.
5. Show that the polynomial $p(x) := x^4 + 7x^3 - 9$ has at least two real roots. Use a calculator to locate these roots to within two decimal places.
6. Let f be continuous on the interval $[0, 1]$ to $\mathbb{R}$ and such that $f(0) = f(1)$. Prove that there exists a point c in $[0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$. [*Hint:* Consider $g(x) = f(x) - f(x + \frac{1}{2})$.] Conclude that there are, at any time, antipodal points on the earth's equator that have the same temperature.
7. Show that the equation $x = \cos x$ has a solution in the interval $[0, \pi/2]$. Use the Bisection Method and a calculator to find an approximate solution of this equation, with error less than 10^{-3}.
8. Show that the function $f(x) := 2 \ln x + \sqrt{x} - 2$ has root in the interval $[1, 2]$, Use the Bisection Method and a calculator to find the root with error less than 10^{-2}.
9. (a) The function $f(x) := (x - 1)(x - 2)(x - 3)(x - 4)(x - 5)$ has five roots in the interval $[0, 7]$. If the Bisection Method is applied on this interval, which of the roots is located?
   (b) Same question for $g(x) := (x - 2)(x - 3)(x - 4)(x - 5)(x - 6)$ on the interval $[0, 7]$.
10. If the Bisection Method is used on an interval of length 1 to find p_n with error $|p_n - c| < 10^{-5}$, determine the least value of n that will assure this accuracy.
11. Let $I := [a, b]$, let $f : I \to \mathbb{R}$ be continuous on I, and assume that $f(a) < 0, f(b) > 0$. Let $W := \{x \in I : f(x) < 0\}$, and let $w := \sup W$. Prove that $f(w) = 0$. (This provides an alternative proof of Theorem 5.3.5.)
12. Let $I := [0, \pi/2]$ and let $f : I \to \mathbb{R}$ be defined by $f(x) := \sup\{x^2, \cos x\}$ for $x \in I$. Show there exists an absolute minimum point $x_0 \in I$ for f on I. Show that x_0 is a solution to the equation $\cos x = x^2$.
13. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim\limits_{x \to -\infty} f = 0$ and $\lim\limits_{x \to \infty} f = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or minimum on $\mathbb{R}$. Give an example to show that both a maximum and a minimum need not be attained.
14. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $\beta \in \mathbb{R}$. Show that if $x_0 \in \mathbb{R}$ is such that $f(x_0) < \beta$, then there exists a δ-neighborhood U of x_0 such that $f(x) < \beta$ for all $x \in U$.

15. Examine which open [respectively, closed] intervals are mapped by $f(x) := x^2$ for $x \in \mathbb{R}$ onto open [respectively, closed] intervals.
16. Examine the mapping of open [respectively, closed] intervals under the functions $g(x) := 1/(x^2 + 1)$ and $h(x) := x^3$ for $x \in \mathbb{R}$.
17. If $f : [0, 1] \to \mathbb{R}$ is continuous and has only rational [respectively, irrational] values, must f be constant? Prove your assertion.
18. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a (not necessarily continuous) function with the property that for every $x \in I$, the function f is bounded on a neighborhood $V_{\delta_x}(x)$ of x (in the sense of Definition 4.2.1). Prove that f is bounded on I.
19. Let $J := (a, b)$ and let $g : J \to \mathbb{R}$ be a continuous function with the property that for every $x \in J$, the function g is bounded on a neighborhood $V_{\delta_x}(x)$ of x. Show by example that g is not necessarily bounded on J.

Section 5.4 Uniform Continuity

Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Definition 5.1.1 states that the following statements are equivalent:

(i) f is continuous at every point $u \in A$;

(ii) given $\varepsilon > 0$ and $u \in A$, there is a $\delta(\varepsilon, u) > 0$ such that for all x such that $x \in A$ and $|x - u| < \delta(\varepsilon, u)$, then $|f(x) - f(u)| < \varepsilon$.

The point we wish to emphasize here is that δ depends, in general, on *both* $\varepsilon > 0$ and $u \in A$. The fact that δ depends on u is a reflection of the fact that the function f may change its values rapidly near certain points and slowly near other points. [For example, consider $f(x) := \sin(1/x)$ for $x > 0$; see Figure 4.1.3.]

Now it often happens that the function f is such that the number δ can be chosen to be independent of the point $u \in A$ and to depend only on ε. For example, if $f(x) := 2x$ for all $x \in \mathbb{R}$, then

$$|f(x) - f(u)| = 2|x - u|,$$

and so we can choose $\delta(\varepsilon, u) := \varepsilon/2$ for all $\varepsilon > 0$ and all $u \in \mathbb{R}$. (Why?)

On the other hand if $g(x) := 1/x$ for $x \in A := \{x \in \mathbb{R} : x > 0\}$, then

$$g(x) - g(u) = \frac{u - x}{ux}. \tag{1}$$

If $u \in A$ is given and if we take

$$\delta(\varepsilon, u) := \inf\{\tfrac{1}{2}u, \tfrac{1}{2}u^2\varepsilon\}, \tag{2}$$

then if $|x - u| < \delta(\varepsilon, u)$, we have $|x - u| < \frac{1}{2}u$ so that $\frac{1}{2}u < x < \frac{3}{2}u$, whence it follows that $1/x < 2/u$. Thus, if $|x - u| < \frac{1}{2}u$, the equality (1) yields the inequality

$$|g(x) - g(u)| \le (2/u^2)|x - u|. \tag{3}$$

Consequently, if $|x - u| < \delta(\varepsilon, u)$, then (2) and (3) imply that

$$|g(x) - g(u)| < (2/u^2)\left(\tfrac{1}{2}u^2\varepsilon\right) = \varepsilon.$$

We have seen that the selection of $\delta(\varepsilon, u)$ by the formula (2) "works" in the sense that it enables us to give a value of δ that will ensure that $|g(x) - g(u)| < \varepsilon$ when $|x - u| < \delta$ and

We shall close this section by stating the important theorem of Weierstrass concerning the approximation of continuous functions by polynomial functions. As would be expected, in order to obtain an approximation within an arbitrarily preassigned $\varepsilon > 0$, we must be prepared to use polynomials of arbitrarily high degree.

5.4.14 Weierstrass Approximation Theorem *Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function. If $\varepsilon > 0$ is given, then there exists a polynomial function p_ε such that $|f(x) - p_\varepsilon(x)| < \varepsilon$ for all $x \in I$.*

There are a number of proofs of this result. Unfortunately, all of them are rather intricate, or employ results that are not yet at our disposal. (A proof can be found in Bartle, ERA, pp. 169–172, which is listed in the References.)

Exercises for Section 5.4

1. Show that the function $f(x) := 1/x$ is uniformly continuous on the set $A := [a, \infty)$, where a is a positive constant.
2. Show that the function $f(x) := 1/x^2$ is uniformly continuous on $A := [1, \infty)$, but that it is not uniformly continuous on $B := (0, \infty)$.
3. Use the Nonuniform Continuity Criterion 5.4.2 to show that the following functions are not uniformly continuous on the given sets.
   (a) $f(x) := x^2, \quad A := [0, \infty)$.
   (b) $g(x) := \sin(1/x), \quad B := (0, \infty)$.
4. Show that the function $f(x) := 1/(1 + x^2)$ for $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.
5. Show that if f and g are uniformly continuous on a subset A of $\mathbb{R}$, then $f + g$ is uniformly continuous on A.
6. Show that if f and g are uniformly continuous on $A \subseteq \mathbb{R}$ and if they are *both* bounded on A, then their product fg is uniformly continuous on A.
7. If $f(x) := x$ and $g(x) := \sin x$, show that both f and g are uniformly continuous on $\mathbb{R}$, but that their product fg is not uniformly continuous on $\mathbb{R}$.
8. Prove that if f and g are each uniformly continuous on $\mathbb{R}$, then the composite function $f \circ g$ is uniformly continuous on $\mathbb{R}$.
9. If f is uniformly continuous on $A \subseteq \mathbb{R}$, and $|f(x)| \geq k > 0$ for all $x \in A$, show that $1/f$ is uniformly continuous on A.
10. Prove that if f is uniformly continuous on a bounded subset A of $\mathbb{R}$, then f is bounded on A.
11. If $g(x) := \sqrt{x}$ for $x \in [0, 1]$, show that there does not exist a constant K such that $|g(x)| \leq K|x|$ for all $x \in [0, 1]$. Conclude that the uniformly continuous g is not a Lipschitz function on $[0, 1]$.
12. Show that if f is continuous on $[0, \infty)$ and uniformly continuous on $[a, \infty)$ for some positive constant a, then f is uniformly continuous on $[0, \infty)$.
13. Let $A \subseteq \mathbb{R}$ and suppose that $f : A \to \mathbb{R}$ has the following property: for each $\varepsilon > 0$ there exists a function $g_\varepsilon : A \to \mathbb{R}$ such that g_ε is uniformly continuous on A and $|f(x) - g_\varepsilon(x)| < \varepsilon$ for all $x \in A$. Prove that f is uniformly continuous on A.
14. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **periodic** on $\mathbb{R}$ if there exists a number $p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Prove that a continuous periodic function on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.

15. Let f and g be Lipschitz functions on A.
   (a) Show that the sum $f + g$ is also a Lipschitz function on A.
   (b) Show that if f and g are bounded on A, then the product fg is a Lipschitz function on A.
   (c) Give an example of a Lipschitz function f on $[0, \infty)$ such that its square f^2 is *not* a Lipschitz function.

16. A function is called *absolutely continuous* on an interval I if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any pair-wise disjoint subintervals $[x_k, y_k]$, $k = 1, 2, \ldots, n$, of I such that $\sum |x_k - y_k| < \delta$ we have $\sum |f(x_k) - f(y_k)| < \varepsilon$. Show that if f satisfies a Lipschitz condition on I, then f is absolutely continuous on I.

Section 5.5 Continuity and Gauges†

We will now introduce some concepts that will be used later—especially in Chapters 7 and 10 on integration theory. However, we wish to introduce the notion of a "gauge" now because of its connection with the study of continuous functions. We first define the notion of a tagged partition of an interval.

5.5.1 Definition A **partition** of an interval $I := [a, b]$ is a collection $\mathcal{P} = \{I_1, \ldots, I_n\}$ of non-overlapping closed intervals whose union is $[a, b]$. We ordinarily denote the intervals by $I_i := [x_{i-1}, x_i]$, where

$$a = x_0 < \cdots < x_{i-1} < x_i < \cdots < x_n = b.$$

The points x_i $(i = 0, \ldots, n)$ are called the **partition points** of $\mathcal{P}$. If a point t_i has been chosen from each interval I_i, for $i = 1, \ldots, n$, then the points t_i are called the **tags** and the set of ordered pairs

$$\dot{\mathcal{P}} = \{(I_1, t_1), \ldots, (I_n, t_n)\}$$

is called a **tagged partition** of I. (The dot signifies that the partition is tagged.)

The "fineness" of a partition $\mathcal{P}$ refers to the lengths of the subintervals in $\mathcal{P}$. Instead of requiring that all subintervals have length less than some specific quantity, it is often useful to allow varying degrees of fineness for different subintervals I_i in $\mathcal{P}$. This is accomplished by the use of a "gauge," which we now define.

5.5.2 Definition A **gauge** on I is a strictly positive function defined on I. If δ is a gauge on I, then a (tagged) partition $\dot{\mathcal{P}}$ is said to be **δ-fine** if

$$(1) \qquad t_i \in I_i \subseteq [t_i - \delta(t_i),\ t_i + \delta(t_i)] \quad \text{for} \quad i = 1, \ldots, n.$$

We note that the notion of δ-fineness requires that the partition be tagged, so we do not need to say "tagged partition" in this case.

A gauge δ on an interval I assigns an interval $[t - \delta(t),\ t + \delta(t)]$ to each point $t \in I$. The δ-fineness of a partition $\dot{\mathcal{P}}$ requires that each subinterval I_i of $\dot{\mathcal{P}}$ is contained in the interval determined by the gauge δ and the tag t_i for that subinterval. This is indicated by the inclusions in (1); see Figure 5.5.1. Note that the length of the subintervals is also controlled by the gauge and the tags; the next lemma reflects that control.

†This section can be omitted on a first reading of this chapter.

Exercises for Section 5.6

1. If $I := [a, b]$ is an interval and $f : I \to \mathbb{R}$ is an increasing function, then the point a [respectively, b] is an absolute minimum [respectively, maximum] point for f on I. If f is strictly increasing, then a is the only absolute minimum point for f on I.
2. If f and g are increasing functions on an interval $I \subseteq \mathbb{R}$, show that $f + g$ is an increasing function on I. If f is also strictly increasing on I, then $f + g$ is strictly increasing on I.
3. Show that both $f(x) := x$ and $g(x) := x - 1$ are strictly increasing on $I := [0, 1]$, but that their product fg is not increasing on I.
4. Show that if f and g are positive increasing functions on an interval I, then their product fg is increasing on I.
5. Show that if $I := [a, b]$ and $f : I \to \mathbb{R}$ is increasing on I, then f is continuous at a if and only if $f(a) = \inf\{f(x) : x \in (a, b]\}$.
6. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be increasing on I. Suppose that $c \in I$ is not an endpoint of I. Show that f is continuous at c if and only if there exists a sequence (x_n) in I such that $x_n < c$ for $n = 1, 3, 5, \ldots$; $x_n > c$ for $n = 2, 4, 6, \ldots$; and such that $c = \lim(x_n)$ and $f(c) = \lim(f(x_n))$.
7. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be increasing on I. If c is not an endpoint of I, show that the jump $j_f(c)$ of f at c is given by $\inf\{f(y) - f(x) : x < c < y,\ x, y \in I\}$.
8. Let f, g be strictly increasing on an interval $I \subseteq \mathbb{R}$ and let $f(x) > g(x)$ for all $x \in I$. If $y \in f(I) \cap g(I)$, show that $f^{-1}(y) < g^{-1}(y)$. [*Hint*: First interpret this statement geometrically.]
9. Let $I := [0, 1]$ and let $f : I \to \mathbb{R}$ be defined by $f(x) := x$ for x rational, and $f(x) := 1 - x$ for x irrational. Show that f is injective on I and that $f(f(x)) = x$ for all $x \in I$. (Hence f is its own inverse function!) Show that f is continuous only at the point $x = \frac{1}{2}$.
10. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on I. If f has an absolute maximum [respectively, minimum] at an interior point c of I, show that f is not injective on I.
11. Let $f(x) := x$ for $x \in [0, 1]$, and $f(x) := 1 + x$ for $x \in (1, 2]$. Show that f and f^{-1} are strictly increasing. Are f and f^{-1} continuous at every point?
12. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function that does not take on any of its values twice and with $f(0) < f(1)$. Show that f is strictly increasing on $[0, 1]$.
13. Let $h : [0, 1] \to \mathbb{R}$ be a function that takes on each of its values exactly twice. Show that h cannot be continuous at every point. [*Hint*: If $c_1 < c_2$ are the points where h attains its supremum, show that $c_1 = 0$, $c_2 = 1$. Now examine the points where h attains its infimum.]
14. Let $x \in \mathbb{R}$, $x > 0$. Show that if $m, p \in \mathbb{Z}$, $n, q \in \mathbb{N}$, and $mq = np$, then $\left(x^{1/n}\right)^m = \left(x^{1/q}\right)^p$.
15. If $x \in \mathbb{R}$, $x > 0$, and if $r, s \in \mathbb{Q}$, show that $x^r x^s = x^{r+s} = x^s x^r$ and $(x^r)^s = x^{rs} = (x^s)^r$.