2050 HW3A

1. Let $r \in (0,1)$. Show that

   (i) Let $\delta > 0$ be such that $1+\delta = \frac{1}{r}$. Show that $(0<) r^n \leq \frac{1}{1+n\delta}$ (Hint: Binomial or Bernoulli).

   (ii) Show that $\lim_n r^n = 0$ (Hint: Squeeze).

2. Let $r \in (0,1)$ and

$$s_n := 1 + r + r^2 + \cdots + r^n \quad (n \in \mathbb{N}).$$

Show that $s_n = \frac{1-r^{n+1}}{1-r}$ $\forall n \in \mathbb{N}$ and

$$\lim_n s_n = \frac{1}{1-r}$$

3. Let $c \in (0,1)$, and let (x_n) be a c-contraction sequence, namely

$$|x_{n+1} - x_n| \leq c|x_n - x_{n-1}|, \quad \forall n \in \mathbb{N} \setminus \{1\}.$$

Show, by MI, that, $\forall n \in \mathbb{N}$

(i) $|x_{n+1} - x_n| \leq c^{n-1} |x_2 - x_1|$

(convention : $c^n = 1$ if $n = 0$),

and so

(ii) $|x_{n+j} - x_{n+j-1}| \leq c^{n+j-2} |x_2 - x_1|, \ \forall j \in \mathbb{N}$

and

(iii) $$|x_{n+j} - x_n| \leq \left(c^{n+j-2} + c^{n+j-3} + \dots + c^{n-1}\right) |x_2 - x_1|$$
$$\leq \frac{c^{n-1}}{1-c} |x_2 - x_1|$$

Consequently, show further that (x_n) is a Cauchy sequence.

4. Let

$$x_{n+1} = 2 + \frac{x_n}{2} \quad \forall n \in \mathbb{N}.$$

Then, for each of the following cases, show that (x_n) converges (and find the value of the limit :

(i) $x_1 = 0$;

(ii) $x_1 = 10$.

(Hint : Can the MCT be applied ?)

5.* Show that $\lim_n \frac{n^7}{(1+\delta)^n} = 0$ (where $\delta > 0$).

Hint (similar to Q1 but expand more terms when apply the Binomial).

6.* Let $x_1 > 0$ and

$$x_{n+1} = x_n + \frac{1}{x_1} \quad \forall\, N.$$

Use two methods below to show that (x_n) does not converge:

(a) Use Q6 of HW 2 ((x_n^2) unbounded)

(b) Use (algebraic computation rules).

7.* Suppose $\lim_n y_n = y$. Show

(i) If $y > 0$ then there exists $N \in \mathbb{N}$ such that

$$0.9 \cdot y < y_n < 2y \quad \forall\, n \geq N.$$

($\frac{9}{10} y < y_n \cdots$).

(ii) If $y \neq 0$ then there exists $N \in \mathbb{N}$ such that

$$0.9 \cdot |y| < |y_n| < 2|y|, \forall\, n \geq N.$$

(iii) Suppose $\lim_n y_n = y$, $y \neq 0$ and $\delta \in (0, |y|)$. Then $\exists\, N \in \mathbb{N}$ s.t.

$$(1-\delta)|y| < |y_n| < \frac{1}{2022} + |y| \quad \forall\, n \geq N.$$

$\hat{3}$. Returning to Q3, show that

$$\sum_{n=1}^{\infty} |x_{n+1} - x_n| \leqslant \sum_{n=1}^{\infty} c^{n-1} |x_2 - x_1| = \frac{1}{1-c} |x_2 - x_1|$$

and hence that $\sum_{n=1}^{\infty} (x_{n+1} - x_n)$ exists in $\mathbb{R}$

that is $\lim_m (x_{m+1} - x_1) \in \mathbb{R}$ and

so $\lim_n x_n$ exists (why?)