Assume Axioms I, II, & III for $\mathbb{R}$.

1. Let $\emptyset \neq X \subseteq \mathbb{R}$ and $\bar{u} \in \mathbb{R}$. Define what is meant by that $\bar{u} = \sup X$, that is u is the smallest upper bound of X by filling the blanks below

(i) $x \leq \bar{u}$ for .... X;

(ii) if $\mathbb{R} \ni w < \bar{u}$ then w ... for .... X.

State the negation (i.e. $\bar{u} \neq \sup X$).

(the def. of $\sup X$ is already provided here & we have not used III).

2. Do Q1 similarly for

$\inf X$ (= the greatest lower bound of X)

3. Show that

$$-\sup X = \inf(-X)$$

provided that either $\sup X$ exists in $\mathbb{R}$ or $\inf(-X)$ exists in $\mathbb{R}$.

4. Let $\emptyset \neq A, B \subseteq \mathbb{R}$ and

$$A+B := \{a+b : a \in A, b \in B\}$$

Show that

$$\sup(A+B) = \sup A + \sup B$$

provided that LHS exists in $\mathbb{R}$ or RHS exists in $\mathbb{R}$ (namely, $\sup A$ & $\sup B$ exist in $\mathbb{R}$).

Note (Warning). The axiom III is needed for this question, e.g.

$\mathbb{Q}$ satisfies I & II but $\exists$ non-empty $A, B \subseteq \mathbb{Q}$ such that $\sup(A+B)$ exists in $\mathbb{Q}$ but $\sup A$, $\sup B$ not exist in $\mathbb{Q}$.

e.g. $A := \{x \in \mathbb{Q} : x < \sqrt{2}\}$

$B := \{x \in \mathbb{Q} : x < 3 - \sqrt{2}\}$

(so $\sup(A+B) = \sup\{x < 3\} = 3 \in \mathbb{Q}$

but $\sup A, \sup B \in \mathbb{R} \setminus \mathbb{Q}$).

5. Let $f, g : D \to \mathbb{R}$ be functions such that

$$\sup\{f(x) : x \in D\}, \text{ and } \sup\{g(x) : x \in D\}$$

exist in $\mathbb{R}$. Show that

$$\sup\{f(x)+g(x) : x \in D\} \leq \sup\{f(x) : x \in D\} + \sup\{g(x) : x \in D\}.$$

and provide a counter-example showing that " $\leq$ " cannot be replaced by " $=$ ".

6.* Let $a, b, x_1 > 0$ (each positive), and

$$x_{n+1} := x_n + \frac{1}{x_n} \quad \forall n \in \mathbb{N}.$$

Show :

(i) $a^2 < b^2$ iff (= if and only if) $a < b$.

(ii) $x_{n+1} > x_n$ and $x_{n+1}^2 > x_n^2$, $\forall n \in \mathbb{N}$.

(iii) $x_{n+1}^2 \geqslant x_{n+1} \cdot x_n = x_n^2 + 1$ and $x_{n+1}^2 > n$, $\forall n \in \mathbb{N}$.

(iv) ~~sequences~~ sequence (x_n^2) ~~and~~ is not bounded and hence (x_n) ~~are~~ is not bounded.

7. Using the combinatory formular

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad (k \leq n, \text{ natural nos.}),$$

Show the Binomial Theorem and Bernoulli Inequality for $a, b > 0$ ($\forall$ $k, n \in \mathbb{N}$ with $k \leq n$) :

(i) $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.

(ii) $(1+a)^n \geqslant \dfrac{n(n-1)\cdots(n-k+1)}{k!} a^k$.

Discuss the situation if the positivity of a, b is dropped.

8.

$\Downarrow$

8*. Let

$$x_{n+1} = 2 + \frac{x_n}{2} \quad \forall\, n \in \mathbb{N}.$$

For each of the cases below, show that (x_n) is monotone (either $\uparrow$ or $\downarrow$, increasing or decreasing not necessarily strictly for the notations and the terminologies.

(i) $x_1 = 1$.

(ii) $x_1 = 10$.

(Hint: try first few terms to get your conjecture).

9.* "Solve" the inequality system:

(#) $4 < |x+2| + |x-1| \leq 5$,

that is, let X consist of all x satisfying the above inequalities, concretely express X.

Hint: Try to remove the absolute value signs.

$$\varphi(t) := |x+2| + |x-1| \quad \forall x \in \mathbb{R}$$

$$= \begin{cases} -(x+2) + (1-x) & \forall x \leq -2 \\ 2+x + (1-x) & \forall -2 < x \leq 1 \\ 2+x + (x-1) & 1 < x \end{cases}$$

$$= \begin{cases} -1 - 2x & x \leq -2 \\ 3 & -2 < x \leq 1 \\ 1+2x & 1 < x \end{cases}.$$

Hence the "solution set" X

(consisting of all x satisfying

$4 < \varphi(x) \leq 5$)

is

$$X = \{x \leq -2 : 4 < -1 - 2x \leq 5\} \cup$$

$$\cup \{x > 1 : 4 < 1 + 2x \leq 5\}$$

$$= \{x \leq -2 : 5 < -2x \leq 6\} \quad -\frac{5}{2} > x \geq -3$$

$$\cup \{x > 1 : 3 < 2x \leq 4\} \quad \frac{3}{2} < x \leq 2$$

$$= [-3, -\tfrac{5}{2}) \cup (\tfrac{3}{2}, 2]$$

10.* Let $\emptyset \neq X \subseteq \mathbb{R}$, bounded above.

Let $\alpha \in \mathbb{R}$ and $\alpha X := \{\alpha x : x \in X\}$. Show that

$$\sup \alpha X = \alpha \cdot \sup X \quad \forall \alpha \geq 0$$

$$\inf(\alpha X) = \alpha \cdot \sup X \quad \forall \alpha < 0$$