MATH2040 Homework 2
Reference Solution

1.5.13. Let V be a vector space over a field of characteristic not equal to two.

(a) Let u and v be distinct vectors in V. Prove that $\{ u, v \}$ is linearly independent if and only if $\{ u+v, u-v \}$ is linearly independent.

(b) Let u, v, and w be distinct vectors in V. Prove that $\{ u, v, w \}$ is linearly independent if and only if $\{ u+v, u+w, v+w \}$ is linearly independent.

Idea: To show that a given (finite) set of vectors is linearly independent, one way is to find constrains on the coefficients for the linear combinations that gives a zero vector, and argue that the coefficients satisfying these constrains must be all zero.

As we are given another set of vectors known to be linearly independent, we can try to relate the linear combinations with these vectors, and solving the system with their linear independence should give a set of constrains on the coefficients.

Solution: Since the characteristic of the scalar field $\mathbb{F}$ is not 2, $2 = 1 + 1 \neq 0$ and $\frac{1}{2}$ exists in $\mathbb{F}$.

(a)

- Suppose $\{ u, v \}$ is linearly independent. Then for scalars $c, d \in \mathbb{F}$ such that $cu + dv = 0$, we must have $c = d = 0$.
  Let $a, b \in \mathbb{F}$ be such that $a(u+v)+b(u-v) = 0$. Then $(a+b)u+(a-b)v = 0$. Since $\{ u, v \}$ is linearly independent, we must have $a + b = a - b = 0$. This implies that $a = \frac{(a+b)+(a-b)}{2} = 0$, $b = \frac{(a+b)-(a-b)}{2} = 0$.
  As a, b are arbitrary, $\{ u+v, u-v \}$ is linearly independent.
- Suppose $\{ u+v, u-v \}$ is linearly independent.
  Let $a, b \in \mathbb{F}$ be such that $au + bv = 0$. Then $0 = \left(\frac{a+b}{2} + \frac{a-b}{2}\right) u + \left(\frac{a+b}{2} - \frac{a-b}{2}\right) v = \frac{a+b}{2}(u+v) + \frac{a-b}{2}(u-v)$. Since $\{ u+v, u-v \}$ is linearly independent, we must have $\frac{a+b}{2} = \frac{a-b}{2} = 0$, so $a = \frac{a+b}{2} + \frac{a-b}{2} = 0$, $b = \frac{a+b}{2} - \frac{a-b}{2} = 0$.
  As a, b are arbitrary, $\{ u, v \}$ is linearly independent.

So $\{ u, v \}$ is linearly independent if and only if $\{ u+v, u-v \}$ is.

(b)

- Suppose $\{ u, v, w \}$ is linearly independent.
  Let $a, b, c \in \mathbb{F}$ be such that $a(u+v)+b(u+w)+c(v+w) = 0$. Then $(a+b)u+(a+c)v+(b+c)w = 0$. Since $\{ u, v, w \}$ is linearly independent, we must have $a + b = b + c = a + c = 0$. This implies that $a = \frac{(a+b)+(a+c)-(b+c)}{2} = 0$, $b = \frac{(a+b)+(b+c)-(a+c)}{2} = 0$, $c = \frac{(a+c)+(b+c)-(a+b)}{2} = 0$.
  As a, b, c are arbitrary, $\{ u+v, u+w, v+w \}$ is linearly independent.
- Suppose $\{ u+v, u+w, v+w \}$ is linearly independent.
  Let $a, b, c \in \mathbb{F}$ be such that $au + bv + cw = 0$. Then

$$
\begin{aligned}
0 &= \left(\frac{a+b-c}{2} + \frac{a-b+c}{2}\right) u + \left(\frac{a+b-c}{2} + \frac{-a+b+c}{2}\right) v + \left(\frac{a-b+c}{2} + \frac{-a+b+c}{2}\right) w \\
&= \frac{a+b-c}{2}(u+v) + \frac{a-b+c}{2}(u+w) + \frac{-a+b+c}{2}(v+w)
\end{aligned}
$$

  Since $\{ u+v, u+w, v+w \}$ is linearly independent, we must have $\frac{a+b-c}{2} = \frac{a-b+c}{2} = \frac{-a+b+c}{2} = 0$, so $a = \frac{a+b-c}{2} + \frac{a-b+c}{2} = 0$, $b = \frac{a+b-c}{2} + \frac{-a+b+c}{2} = 0$, $c = \frac{a-b+c}{2} + \frac{-a+b+c}{2} = 0$.
  As a, b, c are arbitrary, $\{ u, v, w \}$ is linearly independent.

So $\{ u, v, w \}$ is linearly independent if and only if $\{ u+v, u+w, v+w \}$ is.

Note

Some may consider in both parts that the first half of the proposition is easier than the second half, as in the second half the combinations of the coefficients ($\frac{a+b}{2}, \frac{a-b}{2}$ and $\frac{a+b-c}{2}, \frac{a-b+c}{2}, \frac{-a+b+c}{2}$) appear to be pulled out of thin air. However, these combinations can be readily obtained from the first half: if $c = a + b$ and $d = a - b$, then we must have $a = \frac{c+d}{2}$ and $b = \frac{c-d}{2}$ (similar for the second part). In fact, you can show the following statement as an (easy) exercise:

If $A \in M_{n\times n}(\mathbb{F})$ is invertible, then the set of distinct vectors $\{ v_i : i \in \{ 1, \dots, n \} \}$ is linearly independent if and only if $\left\{ \sum_{j=1}^{n} A_{ij} v_j : i \in \{ 1, \dots, n \} \right\}$ is.

It is obvious why we need the characteristic being different from 2.

Some common errors on this problem include (using the first part as example):

- stating $\{ u, v \}$ is linearly independent because $au + bv = 0$ if $a = b = 0$
- showing that $(a+b)u + (a-b)v = 0$ implies $a = b = 0$ and so $a + b = a - b = 0$, and stating that this alone implies the linear independence of $\{ u, v \}$

1.5.15. Let $S = \{ u_1, u_2, \dots, u_n \}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \mathrm{Span}(\{ u_1, u_2, \dots, u_k \})$ for some k $(1 \le k < n)$.

Solution:

- Suppose $u_1 = 0$ or $u_{k+1} \in \mathrm{Span}(\{ u_1, \dots, u_k \})$ for some $k \in \{ 1, \dots, n-1 \}$.
  - If $u_1 = 0$, we have $0 = u_1 \in S$, and so S is linearly dependent
  - If $u_{k+1} \in \mathrm{Span}(\{ u_1, \dots, u_k \})$ for some $k \in \{ 1, \dots, n-1 \}$, we have $u_{k+1} = \sum_{i=1}^{k} a_i u_i$ for some scalars $a_1, \dots, a_k$ and so $\sum_{i=1}^{k} a_i u_i - 1 \cdot u_{k+1} = 0$. Since $u_1, \dots, u_{k+1} \in S$ and the scalars $a_1, \dots, a_k, 1$ are not all zero, S is linearly dependent.

  Thus, S is linearly dependent.
- Suppose S is linearly dependent. Assume that $u_1 \ne 0$. We will show that $u_{k+1} \in \mathrm{Span}(\{ u_1, \dots, u_k \})$ for some $k \in \{ 1, \dots, n-1 \}$.

  Since $u_1 \ne 0$, $\{u_1\}$ is linearly independent. Let $k \le n$ be the largest integer such that $\{ u_1, \dots, u_k \}$ is linearly independent. As $\{u_1\}$ is linearly independent and $S = \{ u_1, \dots, u_n \}$ is linearly dependent, such k exists and $1 \le k < n$.

  By definition, $\{ u_1, \dots, u_k \}$ is linearly independent and $\{ u_1, \dots, u_k, u_{k+1} \}$ is linearly dependent. So there exists scalars $a_1, \dots, a_{k+1}$ not all zero such that $\sum_{i=1}^{k+1} a_i u_i = 0$.

  Suppose $a_{k+1} = 0$. Then we have $0 = \sum_{i=1}^{k+1} a_i u_i = \sum_{i=1}^{k} a_i u_i$ with $a_1, \dots, a_k$ being not all zero. This implies that $\{ u_1, \dots, u_k \}$ is linearly dependent. Contradiction arises. So $a_{k+1} \ne 0$.

  Hence $u_{k+1} = -\frac{1}{a_{k+1}} \sum_{i=1}^{k} a_i u_i = \sum_{i=1}^{k} -\frac{a_i}{a_{k+1}} u_i \in \mathrm{Span}(\{ u_1, \dots, u_k \})$.

So S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \mathrm{Span}(\{ u_1, u_2, \dots, u_k \})$ for some $k \in \{ 1, \dots, n-1 \}$.

Note

In view of Question 1.5.16, this also holds for infinite sets.

1.5.18. Let S be a set of nonzero polynomials in $\mathsf{P}(\mathbb{F})$ such that no two have the same degree. Prove that S is linearly independent.

Solution: The proposition is trivial if S is empty. Hence in the following proof we will assume that $S \ne \emptyset$.

Let $n \in \mathbb{Z}^+$, $p_1, \dots, p_n \in S$ be distinct, and $c_1, \dots, c_n \in \mathbb{F}$ be such that $\sum_{i=1}^{n} c_i p_i = 0$. Suppose there are some $c_i \ne 0$. By removing entries and permuting indices we may assume that $c_i \ne 0$ for all $i \in \{ 1, \dots, n \}$ and $0 \le \deg p_1 \le \ldots \le \deg p_n$. As the degrees are all distinct, we must have $0 \le \deg p_1 < \ldots < \deg p_n$. Then $-\infty = \deg 0 = \deg \sum_{i=1}^{n} c_i p_i = \deg(c_n p_n) = \deg p_n \ge 0$. Contradiction arises. Hence all $c_i = 0$.

Since $n, p_1, \dots, p_n, c_1, \dots, c_n$ are arbitrary, S is linearly independent.

Note

In the proof, we adapt the convention that $\deg 0 = -\infty$ but the proof still works if the convention that $\deg 0 = -1$ is used instead.

1.6.12. Let u, v, and w be distinct vectors of a vector space V. Show that if $\{ u, v, w \}$ is a basis for V, then $\{ u+v+w, v+w, w \}$ is also a basis for V.

Idea: To show that the given set β is a basis of V, we generally need to show

- β is linearly independent;
- β spans the whole space, i.e. $\mathrm{Span}(\beta) = V$

However, since we already have a finite basis α of V, we already know $\dim V$ (which is $|\alpha|$). So by the corollary of Replacement Theorem, as long as $|\beta| = |\alpha|$, we only need to show only one of these two conditions.

Solution: Since $\dim V = |\{ u, v, w \}| = 3 = |\{ u+v+w, v+w, w \}|$, it suffices to show that $\{ u+v+w, v+w, w \}$ is linearly independent.

Let a, b, c be scalars such that $a(u+v+w)+b(v+w)+cw = 0$. Then $au+(a+b)v+(a+b+c)w = 0$. As $\{ u, v, w \}$ is a basis of V, it is linearly independent, and so $a = a+b = a+b+c = 0$. Hence $a = 0$, $b = (a+b)-a = 0$, $c = (a+b+c)-(a+b) = 0$.

As a, b, c are arbitrary, $\{ u+v+w, v+w, w \}$ is linearly independent. So it is a basis of V.

Note

See also Question 1.5.13.

1.6.15. The set of all $n \times n$ matrices having trace equal to zero is a subspace W of $M_{n\times n}(\mathbb{F})$. Find a basis for W. What is the dimension of W?

Idea: To compute the dimension of a subspace, a simple way is to construct explicitly a basis for this subspace, which may be found by working on the constrains that define the subspace. The complexity of the proof usually depends on the choice of the basis.

Solution: We will construct a basis for W. For $i, j \in \{ 1, \dots, n \}$ let $E_{ij} \in M_{n\times n}(\mathbb{F})$ denote the $n \times n$ matrix that the (i, j)-entry is 1 and all other entries are 0, i.e. $(E_{ij})_{kl} = \begin{cases} 1 & \text{if } i = k \text{ and } j = l \\ 0 & \text{otherwise} \end{cases}$ for $k, l \in \{ 1, \dots, n \}$. It is easy to see that $\{ E_{ij} : i, j \in \{ 1, \dots, n \} \}$ is a basis of $M_{n\times n}(\mathbb{F})$.

Let $A \in W \subseteq M_{n\times n}(\mathbb{F})$. Then $A = \sum_{i,j\in\{1,\dots,n\}} A_{ij}E_{ij}$ with scalars $A_{ij} \in \mathbb{F}$. As $A \in W$, $0 = \mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}$ and so $A_{nn} = -\sum_{i=1}^{n-1} A_{ii}$. This implies that

$$A = \sum_{i\neq j} A_{ij}E_{ij} + \sum_{i=1}^{n} A_{ii}E_{ii} = \sum_{i\neq j} A_{ij}E_{ij} + \sum_{i=1}^{n-1} A_{ii}E_{ii} - \left(\sum_{i=1}^{n-1} A_{ii}\right) E_{nn} = \sum_{i\neq j} A_{ij}E_{ij} + \sum_{i=1}^{n-1} A_{ii}(E_{ii} - E_{nn}) \in \mathrm{Span}(\beta)$$

with $\beta = \{ E_{ij} : i \neq j \} \cup \{ E_{ii} - E_{nn} : i \in \{ 1, \dots, n-1 \} \}$. As A is arbitrary, $W \subseteq \mathrm{Span}(\beta)$. It is also easy to see that $\beta \subseteq W$, so $\mathrm{Span}(\beta) \subseteq W$, which implies that β spans W.

It then suffices to show that β is linearly independent. Let $A_{ij} \in \mathbb{F}$ for $i, j \in \{ 1, \dots, n \}$ with $i \neq j$ and $B_i \in \mathbb{F}$ for $i \in \{ 1, \dots, n-1 \}$ be scalars such that $\sum_{i\neq j} A_{ij}E_{ij} + \sum_{i=1}^{n-1} B_i(E_{ii} - E_{nn}) = 0_{n\times n}$. Then $\sum_{i\neq j} A_{ij}E_{ij} + \sum_{i=1}^{n-1} B_iE_{ii} - \left(\sum_{i=1}^{n-1} B_i\right) E_{nn} = 0_{n\times n}$. Since $\{ E_{ij} : i, j \in \{ 1, \dots, n \} \}$ is a basis of $M_{n\times n}(\mathbb{F})$, we have $A_{ij} = 0$ for $i \neq j$ and $B_i = 0$ for $i \in \{ 1, \dots, n-1 \}$. This implies that β is linearly independent.

Since β is linearly independent and spans W, β is a basis of W, and so $\dim(W) = |\beta| = n^2 - 1$.

1.6.23. Let $v_1, v_2, \dots, v_k, v$ be vectors in a vector space V, and define $W_1 = \mathrm{Span}(\{ v_1, v_2, \dots, v_k \})$, and $W_2 = \mathrm{Span}(\{ v_1, v_2, \dots, v_k, v \})$.

(a) Find necessary and sufficient conditions on v such that $\dim(W_1) = \dim(W_2)$.

(b) State and prove a relationship involving $\dim(W_1)$ and $\dim(W_2)$ in the case that $\dim(W_1) \neq \dim(W_2)$.

Solution:

(a) The sufficient and necessary condition is that $v \in W_1$.

Since $\{ v_1, \dots, v_k \} \subseteq \{ v_1, \dots, v_k, v \}$, we always have $W_1 = \text{Span}(\{ v_1, v_2, \dots, v_k \}) \subseteq \text{Span}(\{ v_1, v_2, \dots, v_k, v \}) = W_2$.

- Suppose $\dim(W_1) = \dim(W_2)$. Since $W_1 \subseteq W_2$, we have $W_1 = W_2$, and thus $v \in W_2 = W_1$.
- Suppose $v \in W_1$. The $\{ v_1, \dots, v_k, v \} \subseteq W_1$, so $W_2 = \text{Span}(\{ v_1, v_2, \dots, v_k, v \}) \subseteq W_1$. So $W_1 = W_2$ and thus $\dim(W_1) = \dim(W_2)$.

Hence $\dim(W_1) = \dim(W_2)$ if and only if $v \in W_1$.

(b) If $\dim(W_1) \neq \dim(W_2)$, we have $\dim(W_2) = \dim(W_1) + 1$.

Suppose $\dim(W_1) \neq \dim(W_2)$. Then by the previous part, $v \notin W_1$. Let $\beta \subseteq W_1$ be a basis of W_1. Then β is linearly independent. As $v \notin W_1 = \text{Span}(\beta)$, $\beta \cup \{v\}$ is also linearly independent. Moreover, $W_2 = \text{Span}(\{ v_1, v_2, \dots, v_k, v \}) = \text{Span}(\{ v_1, v_2, \dots, v_k \}) + \text{Span}(\{ v \}) = W_1 + \text{Span}(\{ v \}) = \text{Span}(\beta) + \text{Span}(\{ v \}) = \text{Span}(\beta \cup \{v\})$, so $\beta \cup \{v\}$ spans W_2. Thus $\beta \cup \{v\}$ is a basis of W_2, and so $\dim(W_2) = | \beta \cup \{v\} | = | \beta | + 1 = \dim(W_1) + 1$.

Note

Stating only $\dim(W_1) < \dim(W_2)$, $\dim(W_1) \leq \dim(W_2)$, $\dim(W_1) \leq k$, $\dim(W_2) \leq k + 1$, or any combination of $\dim(W_1), \dim(W_2)$ being an integer will be counted as incorrect answers as these are too trivial.

1.6.26. For a fixed $a \in \mathbb{R}$, determine the dimension of the subspace of $\mathsf{P}_n(\mathbb{R})$ defined by $\{ f \in \mathsf{P}_n(\mathbb{R}) : f(a) = 0 \}$.

Solution: Denote the subspace as V.

We first consider what property polynomials in V has.

Let $f \in V$. Then $f(a) = 0$, and so by factor theorem / remainder theorem, $f(x) = (x - a)g(x)$ for some polynomial $g \in \mathsf{P}(\mathbb{R})$. Since $\deg f \leq n$, we must have $\deg g \leq \deg f - 1 \leq n - 1$ and so $g \in \mathsf{P}_{n-1}(\mathbb{R})$. This implies that $g(x) = \sum_{i=0}^{n-1} c_i x^i$ for some $c_1, \dots, c_{n-1} \in \mathbb{R}$, and $f(x) = (x - a)g(x) = \sum_{i=0}^{n-1} c_i (x - a) x^i \in \text{Span}(\beta)$ with $\beta = \left\{ (x - a), (x - a)x, \dots, (x - a)x^{n-1} \right\}$ being a set of nonzero polynomials.

Furthermore, for each $g \in \beta$ we have $g(a) = 0$, so $\beta \subseteq V$ and thus $\text{Span}(\beta) \subseteq V$, which implies that β spans V.

It remains to show that β is linearly independent, as this would implies that $\dim(V) = | \beta | = n$. However, note that $\deg((x - a)x^i) = i + 1$ for each $i \in \{ 0, \dots, n - 1 \}$, and so no two polynomials in β have the same degree. By Question 1.5.18, β is linearly independent.

Thus $\dim(V) = n$.

Note

Another popular choice for basis of V is $\left\{ x^i - a^i : i \in \{ 1, \dots, n \} \right\}$, for which the proof does not require using factor theorem.

1.6.29. (a) Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$, is finite-dimensional, and $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

(b) Let W_1 and W_2 be finite-dimensional subspaces of a vector space V, and let $V = W_1 + W_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if $\dim(V) = \dim(W_1) + \dim(W_2)$.

Solution:

(a) Since $W_1, W_2 \subseteq V$, we have $W_1 + W_2 \subseteq V$ and $W_1 \cap W_2 \subseteq V$. As V is finite-dimensional, so are $W_1 + W_2$ and $W_1 \cap W_2$.

let $\gamma \subseteq W_1 \cap W_2$ be a basis of $W_1 \cap W_2$. By Extension Theorem, we may extend γ to a basis $\beta_1 = \gamma \cup \alpha_1$ of W_1 and a basis $\beta_2 = \gamma \cup \alpha_2$ of W_2 with $\alpha_1 \subseteq W_1$, $\alpha_2 \subseteq W_2$ respectively and $\gamma \cap \alpha_1 = \gamma \cap \alpha_2 = \emptyset$.

Suppose $\alpha_1 \cap \alpha_2 \neq \emptyset$. Then there exists $v \in \alpha_1 \cap \alpha_2 \subseteq W_1 \cap W_2 = \text{Span}(\gamma)$. This implies that $\gamma \cup \{v\}$ is linearly dependent, and so is $\beta_1 = \gamma \cup \alpha_1 \supseteq \gamma \cup \{v\}$ (and equivalently β_2). Contradiction arises. Hence $\alpha_1 \cap \alpha_2 = \emptyset$. This implies that $\gamma, \alpha_1, \alpha_2$ are disjoint.

To show the proposition, we will show that $\beta = \gamma \cup \alpha_1 \cup \alpha_2$ is a basis of $W_1 + W_2$.

Since $\beta = \gamma \cup \alpha_1 \cup \alpha_2 = (\gamma \cup \alpha_1) \cup (\gamma \cup \alpha_2) = \beta_1 \cup \beta_2$, we have $W_1 + W_2 = \text{Span}(\ \beta_1\ ) + \text{Span}(\ \beta_2\ ) = \text{Span}(\ \beta_1 \cup \beta_2\ ) = \text{Span}(\ \beta\ )$. So β spans $W_1 + W_2$.

Since $W_1, W_2, W_1 \cap W_2$ are all finite-dimensional, we may assume that $\alpha_1 = \{\ w_1, \ldots, w_n\ \}$, $\alpha_2 = \{\ w'_1, \ldots, w'_m\ \}$, $\gamma = \{\ v_1, \ldots, v_p\ \}$ for some $n, m, p \in \mathbb{N}$. Let $a_1, \ldots, a_n, a'_1, \ldots, a'_m, b_1, \ldots, b_p \in \mathbb{F}$ be such that $\sum_{i=1}^n a_i w_i + \sum_{i=1}^m a'_i w'_i + \sum_{i=1}^p b_i v_i = 0$. Then $\sum_{i=1}^n a_i w_i + \sum_{i=1}^p b_i v_i = -\sum_{i=1}^m a'_i w'_i \in \text{Span}(\ \beta_1\ ) \cap \text{Span}(\ \alpha_2\ ) \subseteq W_1 \cap W_2$, so $\sum_{i=1}^n a_i w_i + \sum_{i=1}^p b_i v_i = \sum_{i=1}^p c_i v_i$ for some scalars $c_1, \ldots, c_p$. This implies that $\sum_{i=1}^n a_i w_i + \sum_{i=1}^p (b_i - c_i) v_i = 0$. By the linear independence of β_1, $a_1 = \ldots = a_n = 0$ and so $\sum_{i=1}^m a'_i w'_i + \sum_{i=1}^p b_i v_i = 0$. By the linear independence of β_2, $a'_1 = \ldots = a'_m = b_1 = \ldots = b_p = 0$. This implies that β is linearly independent.

In particular, β is a basis of $W_1 + W_2$.

Thus, we have $\dim(W_1 + W_2) = |\ \beta\ | = |\ \gamma\ | + |\ \alpha_1\ | + |\ \alpha_2\ | = \dim(W_1 \cap W_2) + (\dim(W_1) - \dim(W_1 \cap W_2)) + (\dim(W_2) - \dim(W_1 \cap W_2)) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

(b) Suppose $V = W_1 \oplus W_2$. Then $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. So by the previous part, $\dim(V) = \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2)$.

Suppose $\dim(W_1) + \dim(W_2) = \dim(V)$. Then by the previous part, $\dim(V) = \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = \dim(V) - \dim(W_1 \cap W_2)$. This implies that $\dim(W_1 \cap W_2) = 0$ and so $W_1 \cap W_2 = \{0\}$. As $W_1 + W_2 = V$, we have $V = W_1 \oplus W_2$.

Therefore $V = W_1 \oplus W_2$ if and only if $\dim(V) = \dim(W_1) + \dim(W_2)$.

Note

Please send us an email if you find an easy proof on part (a) that does not involve solving for the witnessing linear combination.

Generalizing part (a) to 3 (or more) subspaces is an interesting exercise, and we encourage you to work on it.

The argument for part (a) still works (with appropriate modifications) when V is not necessarily finite-dimensional, although you then only have $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2)$.

1.6.30. Let

$$V = M_{2\times 2}(\mathbb{F}), \qquad W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in V \ :\ a, b, c \in \mathbb{F} \right\}$$

and

$$W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \in V \ :\ a, b \in \mathbb{F} \right\}$$

Prove that W_1 and W_2 are subspaces of V, and find the dimensions of W_1, W_2, $W_1 + W_2$, and $W_1 \cap W_2$.

Solution:

- It is easy to see that the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in W_1 and in W_2.
  - Let $A_1, A_2 \in W_1$ and $\gamma \in \mathbb{F}$. Then $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & a_1 \end{pmatrix}, A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & a_2 \end{pmatrix}$ for some $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{F}$. So $W_1 + W_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & a_1 + a_2 \end{pmatrix} \in W_1$ and $\gamma A_1 = \begin{pmatrix} \gamma a_1 & \gamma b_1 \\ \gamma c_1 & \gamma a_1 \end{pmatrix} \in W_1$. As A_1, A_2, γ is arbitrary, W_1 is a subspace.
  - Let $A_1, A_2 \in W_2$ and $\gamma \in \mathbb{F}$. Then $A_1 = \begin{pmatrix} 0 & a_1 \\ -a_1 & b_1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & a_2 \\ -a_2 & b_2 \end{pmatrix}$ for some $a_1, a_2, b_1, b_2 \in \mathbb{F}$. So $W_1 + W_2 = \begin{pmatrix} 0 & a_1 + a_2 \\ -(a_1 + a_2) & b_1 + b_2 \end{pmatrix} \in W_2$ and $\gamma A_1 = \begin{pmatrix} 0 & \gamma a_1 \\ -\gamma a_1 & \gamma b_1 \end{pmatrix} \in W_2$. As A_1, A_2, γ is arbitrary, W_2 is a subspace.
- To find the dimensions of the subspaces, we find a basis for each of them.
  - Let $\beta_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$. Then $\beta_1 \subseteq W_1$, so $\text{Span}(\ \beta_1\ ) \subseteq W_1$.

    Let $A \in W_1$. Then $A = \begin{pmatrix} a & b \\ c & a \end{pmatrix} = a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \text{Span}(\ \beta_1\ )$. As A is arbitrary, $\text{Span}(\ \beta_1\ ) = W_1$.

Let $a, b, c \in \mathbb{F}$ be such that $a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and so $a = b = c = 0$. This implies that β_1 is linearly independent.
Hence β_1 is a basis of W_1 and so $\dim(W_1) = |\ \beta_1\ | = 3$.

– Let $\beta_2 = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Then $\beta_2 \subseteq W_2$, so $\text{Span}(\ \beta_2\ ) \subseteq W_2$.
Let $A \in W_2$. Then $A = \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} = a \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \text{Span}(\ \beta_2\ )$. As A is arbitrary, $\text{Span}(\ \beta_2\ ) = W_2$.
Let $a, b \in \mathbb{F}$ be such that $a \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and so $a = b = 0$. This implies that β_2 is linearly independent.
Hence β_2 is a basis of W_2 and so $\dim(W_2) = |\ \beta_2\ | = 2$.

– It is easy to see that $W_1 + W_2 \subseteq V$.
Let $A \in V$. Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some $a, b, c, d \in \mathbb{F}$. Then $A = \begin{pmatrix} a & 0 \\ c+b & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & d-a \end{pmatrix}$ with $\begin{pmatrix} a & 0 \\ c+b & a \end{pmatrix} \in W_1$, $\begin{pmatrix} 0 & b \\ -b & d-a \end{pmatrix} \in W_2$, so $A \in W_1 + W_2$. As A is arbitrary, $V \subseteq W_1 + W_2$. Hence $V = W_1 + W_2$ and so $\dim(W_1 + W_2) = \dim(V) = 4$.

– Let $A \in W_1 \cap W_2$. Then $A = \begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} 0 & d \\ -d & e \end{pmatrix}$ for some $a, b, c, d, e \in \mathbb{F}$. So $a = e = 0$, $b = d$, $c = -d$, hence $A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \in \text{Span}\left(\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \right)$.
It is easy to see that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in W_1$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in W_2$, so $\text{Span}\left(\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \right) \in W_1 \cap W_2$. This implies that $\text{Span}\left(\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \right) = W_1 \cap W_2$.
As $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is not the zero matrix, $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ is linearly independent and so is a basis of $W_1 \cap W_2$. Thus $\dim(W_1 \cap W_2) = 1$

2.1.5. Prove that T is a linear transformation, and find bases for both $\mathsf{N}(\ T\ )$ and $\mathsf{R}(\ T\ )$. Then compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is one-to-one or onto.

$$T : \mathsf{P}_2(\mathbb{R}) \to \mathsf{P}_3(\mathbb{R}) \text{ defined by } T(f(x)) = xf(x) + f'(x)$$

Solution:

(a) Let $f, g \in \mathsf{P}_2(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Then

- $T(f+g) = x(f+g)(x)+(f+g)'(x) = x(f(x)+g(x))+(f'(x)+g'(x)) = (xf(x)+f'(x))+(xg(x)+g'(x)) = T(f)+T(g)$
- $T(\alpha f) = x(\alpha f)(x) + (\alpha f)'(x) = \alpha(xf(x) + f'(x)) = \alpha T(f)$

As f, g, α are arbitrary, T is linear.

(b) Let $f \in \mathsf{N}(\ T\ ) \subseteq \mathsf{P}_2(\mathbb{R})$. Then $f = a + bx + cx^2$ for some $a, b, c \in \mathsf{P}_2(\mathbb{R})$. Since $f \in \mathsf{N}(\ T\ )$, $0 = T(f) = xf(x) + f'(x) = x(a + bx + cx^2) + (b + 2cx) = b + (a + 2c)x + bx^2 + cx^3$, so $b = a + 2c = c = 0$ and thus $a = b = c = 0$, $f = 0$. This implies that $\mathsf{N}(\ T\ ) = \{0\}$ and a basis for $\mathsf{N}(\ T\ )$ is $\emptyset$.
Let $\beta = \left\{\ T(1), T(x), T(x^2)\ \right\} = \left\{\ x, 1 + x^2, 2x + x^3\ \right\} \subseteq \mathsf{P}_3(\mathbb{R})$. Since $\beta \subseteq \mathsf{R}(\ T\ )$, we have $\text{Span}(\ \beta\ ) \subseteq \mathsf{R}(\ T\ )$. Also, for each $f \in \mathsf{P}_2(\mathbb{R})$, we have $f = a + bx + cx^2$ for some $a, b, c \in \mathbb{R}$ and so $T(f) = aT(1) + bT(x) + cT(x^2) \in \text{Span}(\ \beta\ )$. As $f \in \mathsf{P}_2(\mathbb{R})$ is arbitrary, $\mathsf{R}(\ T\ ) \subseteq \text{Span}(\ \beta\ )$. This implies that β spans $\mathsf{R}(\ T\ )$.
Since β does not contain the zero polynomial and no two polynomials in β has the same degree, by the result of Question 1.5.18, β is linearly independent. Hence β is a basis of $\mathsf{R}(\ T\ )$.

(c) Since $\mathsf{N}(\ T\ ) = \{0\}$, $\text{nullity}\, T = \dim(\mathsf{N}(\ T\ )) = 0$.
Since β is a basis of $\mathsf{R}(\ T\ )$, $\text{rank}\, T = |\ \beta\ | = 3$.

(d) By the previous part, $\text{nullity}\, T + \text{rank}\, T = 0 + 3 = 3$ and $\dim(\mathsf{P}_2(\mathbb{R})) = 3$, so $\text{nullity}\, T + \text{rank}\, T = \dim(\mathsf{P}_2(\mathbb{R}))$, which is consistent with the dimension theorem.

(e) Since $\mathsf{N}(T) = \{0\}$, T is one-to-one.

Since $\dim(\mathsf{R}(T)) = 3 < 4 = \dim(\mathsf{P}_3(\mathbb{R}))$, we have $\mathsf{R}(T) \neq \mathsf{P}_3(\mathbb{R})$ and so T is not onto.

Note

You can also see that T is one-to-one by observing that $\deg T(f) = 1 + \deg f$ for nonzero polynomial f.

2.1.14. Let V and W be vector spaces and $T : V \to W$ be linear.

(a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

(b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

(c) Suppose $\beta = \{ v_1, v_2, \ldots, v_n \}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{ T(v_1), T(v_2), \ldots, T(v_n) \}$ is a basis for W.

Solution:

(a) Suppose T is one-to-one. Let $S \subseteq V$ be a linearly independent set. We want to show that $T(S)$ is also linearly independent.

If $T(S) = \emptyset$, the proposition is trivial, so we may assume that $T(S) \neq \emptyset$. Let $n \in \mathbb{Z}^+$, $w_1, \ldots, w_n \in T(S)$ be distinct, and $c_1, \ldots, c_n \in \mathbb{F}$ be such that $\sum_{i=1}^n c_i w_i = 0$. Since $w_i \in T(S)$ for all i, there exists $v_1, \ldots, v_n \in S$ such that $w_i = T(v_i)$ for all i. So $0 = \sum_{i=1}^n c_i w_i = \sum_{i=1}^n c_i T(v_i) = T(\sum_{i=1}^n c_i v_i)$. Since T is one-to-one, $\sum_{i=1}^n c_i v_i = 0$. Since S is linearly independent, $c_1 = \ldots = c_n = 0$. This implies that $T(S)$ is linearly independent.

Suppose T maps linearly independent subsets of V to linearly independent subsets of W. Let $v \in \mathsf{N}(T)$. If $v \neq 0$, we must have that $\{v\}$ is a linearly independent subset of V, and so $T\{v\} = \{T(v)\} = \{0\}$ is also linearly independent, which is a contradiction. So $v = 0$. This implies that $\mathsf{N}(T) = \{0\}$ and so T is one-to-one.

(b) Suppose S is linearly independent. By the previous part, $T(S)$ is linearly independent.

Suppose $T(S)$ is linearly independent. The proposition is again trivial if $S = \emptyset$. So we may assume that $S \neq \emptyset$. As $T(S)$ is linearly independent, we must have $0 \notin T(S)$.

Let $n \in \mathbb{Z}^+$, $v_1, \ldots, v_n$ be district, $c_1, \ldots, c_n$ be scalars such that $\sum_{i=1}^n c_i v_i = 0$. Then $0 = T(0) = T(\sum_{i=1}^n c_i v_i) = \sum_{i=1}^n c_i T(v_i)$. As $v_1, \ldots, v_n$ are distinct and T is one-to-one, $T(v_1), \ldots, T(v_n) \in T(S)$ are also nonzero distinct. Since $T(S)$ is linearly independent, this implies that $c_1 = \ldots = c_n = 0$. As $n, v_1, \ldots, v_n, c_1, \ldots, c_n$ are arbitrary, this implies that S is linearly independent.

Therefore, S is linearly independent if and only if $T(S)$ is.

(c) Since β is a basis of V, it is linearly independent. By the previous part, $T(\beta)$ is also linearly independent. To show that $T(\beta)$ is a basis of W, it then suffices to show that $T(\beta)$ spans W. Since $\beta \subseteq V$, we trivially have $T(\beta) \subseteq T(V) \subseteq W$ and so $\text{Span}(\, T(\beta)\,) \subseteq W$.

Let $w \in W$. As T is onto, there exists $v \in V$ such that $w = T(v)$. As β is a basis of V, there exists scalars $c_1, \ldots, c_n$ such that $v = \sum_{i=1}^n c_i v_i$. So $w = T(v) = T(\sum_{i=1}^n c_i v_i) = \sum_{i=1}^n c_i T(v_i) \in \text{Span}(\, T(\beta)\,)$. As w is arbitrary, $W \subseteq \text{Span}(\, T(\beta)\,)$. So $T(\beta)$ spans W.

Therefore $T(\beta)$ is a basis of W.

2.1.18. Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathsf{N}(T) = \mathsf{R}(T)$.

Solution: Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$ and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$ for $x \in \mathbb{R}^2$. It is easy to see that T is linear, and $\mathsf{N}(T) = \text{Span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \right) = \mathsf{R}(T)$.

Note

$\mathsf{R}(T) = \mathsf{N}(T)$ implies that $T^2 = 0$. Note that $T \neq 0$.

2.1.21. Let V be the vector space of sequences. Define the functions $T, U : V \to V$ by

$$T(a_1, a_2, \ldots) = (a_2, a_3, \ldots) \text{ and } U(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)$$

(a) Prove that T and U are linear.

(b) Prove that T is onto, but not one-to-one.

(c) Prove that U is one-to-one, but not onto.

Solution:

(a) Let $a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots) \in V$ and $\gamma \in \mathbb{F}$ be a scalar. Then

- $T(a + b) = T(a_1 + b_1, a_2 + b_2, \ldots) = (a_2 + b_2, a_3 + b_3, \ldots) = (a_2, a_3, \ldots) + (b_2, b_3, \ldots) = T(a) + T(b)$
- $U(a + b) = U(a_1 + b_1, a_2 + b_2, \ldots) = (0, a_1 + b_1, a_2 + b_2, \ldots) = (0, a_1, a_2, \ldots) + (0, b_1, b_2, \ldots) = U(a) + U(b)$
- $T(\gamma a) = T(\gamma a_1, \gamma a_2, \ldots) = (\gamma a_2, \gamma a_3, \ldots) = \gamma(a_2, a_3, \ldots) = \gamma T(a)$
- $U(\gamma a) = U(\gamma a_1, \gamma a_2, \ldots) = (0, \gamma a_1, \gamma a_2, \ldots) = \gamma(0, a_1, a_2, \ldots) = \gamma U(a)$

As a, b, γ are arbitrary, T, U are linear

(b) Let $a = (a_1, a_2, \ldots) \in V$. Then $a = (a_1, a_2, \ldots) = T(0, a_1, a_2, \ldots) \in \mathsf{R}(T)$. As a is arbitrary, this implies that T is onto.

Since $T(0, 0, 0, \ldots) = (0, 0, \ldots) = T(1, 0, 0, \ldots)$ and $(0, 0, 0, \ldots) \neq (1, 0, 0, \ldots)$, T is not one-to-one.

(c) Let $a = (a_1, a_2, \ldots) \in \mathsf{N}(U)$. Then $(0, 0, 0, \ldots) = 0 = U(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)$. This implies that $a_i = 0$ for all i. As a is arbitrary and that U is linear, this implies that U is one-to-one.

Since $U(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots) \neq (1, 0, 0, \ldots)$ for all $(a_1, a_2, \ldots) \in V$, $(1, 0, 0, \ldots) \notin \mathsf{R}(U)$ and so U is not onto.

Note

$TU = \mathrm{Id}_V$ is the identity map on V but $UT \neq \mathrm{Id}_V$.

2.1.22. Let $T : \mathbb{R} \to \mathbb{R}$ be linear. Show that there exist scalars a, b, and c such that $T(x, y, z) = ax + by + cz$ for all $(x, y, z) \in \mathbb{R}^3$. Can you generalize this result for $T : \mathbb{F}^n \to \mathbb{F}$? State and prove an analogous result for $T : \mathbb{F}^n \to \mathbb{F}^m$.

Solution:

(a) Let $a = T(1, 0, 0), b = T(0, 1, 0), c = T(0, 0, 1) \in \mathbb{R}$. Then for all $(x, y, z) \in \mathbb{R}^3$, we have $(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$ and so $T(x, y, z) = T(x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)) = xT(1, 0, 0) + yT(0, 1, 0) + zT(0, 0, 1) = ax + by + cz$.

(b) For linear $T : \mathbb{F}^n \to \mathbb{F}$, there exists $a_1, \ldots, a_n \in \mathbb{F}$ such that $T(x_1, \ldots, x_n) = \sum_{i=1}^n a_i x_i$ for all $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

(c) For linear $T : \mathbb{F}^n \to \mathbb{F}^m$, there exists $a_{ij} \in \mathbb{F}$ for $i \in \{ 1, \ldots, n \}$, $j \in \{ 1, \ldots, m \}$ such that $T(x_1, \ldots, x_n) = \left(\sum_{i=1}^n a_{1i} x_i, \ldots, \sum_{i=1}^n a_{mi} x_i \right)$ for all $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

For each $i \in \{ 1, \ldots, n \}$ let $e_i \in \mathbb{F}^n$ be the vector where the ith entry is 1 and all other entries are 0, i.e. $(e_i)_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$ for all $i \in \{ 1, \ldots, n \}$. It is easy to see that $\{ e_1, \ldots, e_n \}$ is a basis of $\mathbb{F}^n$. For each $i \in \{ 1, \ldots, n \}$ let $a_{i1}, \ldots, a_{im} \in \mathbb{F}$ be such that $T(e_i) = (a_{i1}, \ldots, a_{im})$. Then for all $(x_1, \ldots, x_n) \in \mathbb{F}^n$ we have $T(x_1, \ldots, x_n) = T(\sum_{i=1}^n x_i e_i) = \sum_{i=1}^n x_i T(e_i) = \sum_{i=1}^n x_i (a_{i1}, \ldots, a_{im}) = \left(\sum_{i=1}^n a_{1i} x_i, \ldots, \sum_{i=1}^n a_{mi} x_i \right)$.

Note

Part (c) is an easy generalization of part (b) if you have shown the following lemma: if U, V, W are vector spaces over the same scalar field, $\pi_U : U \times W \to U$, $\pi_W : U \times W \to W$ are the projections to the first component and to the second component respectively, then $T : V \to U \times W$ is linear if and only if both $\pi_U T, \pi_W T$ are.

2.1.37. Prove that if V and W are vector spaces over the field of rational numbers, then any additive function from V into W is a linear transformation.

Solution: Let $T : V \to W$ be an additive function. By definition, to show that T is linear, it suffices to show that T is homogeneous, that is, $T(qv) = qT(v)$ for all $q \in \mathbb{Q}$.

We first consider what property an additive map has. Fix $v \in V$.

Trivially, $T(1 \cdot v) = T(v) = 1 \cdot T(v)$. Suppose we have $T(k \cdot v) = k \cdot T(v)$ for some $k \in \mathbb{Z}^+$. Then $T((k+1) \cdot v) = T(k \cdot v + 1 \cdot v) = T(k \cdot v) + T(1 \cdot v) = k \cdot T(v) + 1 \cdot T(v) = (k+1) \cdot T(v)$. So by induction, $T(n \cdot v) = n \cdot T(v)$ for all $n \in \mathbb{Z}^+$.

As $T(v) = T(v + 0_V) = T(v) + T(0_V)$, we have $T(0_V) = 0_W = 0 \cdot T(v)$. Also, for all $n \in \mathbb{Z}^+$, $0_W = T(0_V) = T\,(\,(n \cdot v) + ((-n) \cdot v)\,) = T(n \cdot v) + T((-n) \cdot v) = n \cdot T(v) + T((-n) \cdot v)$, we have $T((-n) \cdot v) = -(n \cdot T(v)) = (-n) \cdot T(v)$.

Thus, $T(n \cdot v) = n \cdot T(v)$ for all $n \in \mathbb{Z}$. As v is arbitrary, this holds for all $v \in V$. Thus for all $n \in \mathbb{Z}^+$ and all $v \in V$, $T(v) = T\left(\,n \cdot (\frac{1}{n} \cdot v)\,\right) = n \cdot T(\frac{1}{n} \cdot v)$ and so $T(\frac{1}{n} \cdot v) = \frac{1}{n} \cdot T(v)$.

Let $q \in \mathbb{Q}$. Then there exists $n \in \mathbb{Z}$, $m \in \mathbb{Z}^+$ such that $q = \frac{n}{m}$. Then for all $v \in V$, $T(q \cdot v) = T(\frac{n}{m} \cdot v) = T\left(\,n \cdot (\frac{1}{m} \cdot v)\,\right) = n \cdot T(\frac{1}{m} \cdot v) = (n \cdot \frac{1}{m}) \cdot T(v) = q \cdot T(v)$.

As q is arbitrary, T is homogeneous and so is linear.

Note

This proposition holds as $\mathbb{Q}$ is the field of fraction of $\mathbb{N}$, and additivity implies the homogeneity on $\mathbb{N}$. See also the next question (Question 2.1.38).

2.1.38. Let $T : \mathbb{C} \to \mathbb{C}$ be the function defined by $T(z) = \overline{z}$. Prove that T is additive but not linear.

Solution: Let $x, y \in \mathbb{C}$. Then $T(x + y) = \overline{x + y} = \overline{x} + \overline{y} = T(x) + T(y)$. As x, y are arbitrary, T is additive.

As $T(1) = \overline{1} = 1$ and $T(i) = \overline{i} = -i$, we have $T(i \cdot 1) = T(i) = -i \neq i = i \cdot T(1)$. So T is not linear.

Note

Note that T is $\mathbb{R}$-linear but not $\mathbb{C}$-linear, whereas $\mathbb{C}$ is (usually) equipped with $\mathbb{C}$-scalars. If $\mathbb{C}$ is equipped with $\mathbb{R}$-scalars (that is, as a real vector space instead of a complex vector space, see Question 1.6.28), T would be linear.

Practice Problems

1.5.1. Label the following statements as true or false.

(a) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.

(b) Any set containing the zero vector is linearly dependent.

(c) The empty set is linearly dependent.

(d) Subsets of linearly dependent sets are linearly dependent.

(e) Subsets of linearly independent sets are linearly independent.

(f) If $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$ and $x_1, x_2, \ldots, x_n$ are linearly independent, then all the scalars a_i are zero.

Solution:

(a) False

(b) True

(c) False. You can verify that there is no nontrivial linear relation between elements of the empty set (as there is no element in the empty set and so no such linear relation exists).

(d) False

(e) True

(f) True

1.5.2. Determine whether the following sets are linearly dependent. or linearly independent.

(a) $\left\{ \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$

(b) $\left\{ \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$

(c) $\{ x^3 - x, 2x^2 + 4, -2x^3 + 3x^2 + 2x + 6 \}$ in $\mathsf{P}_3(\mathbb{R})$

(d) $\{ (1, -1, 2), (1, -2, 1), (1, 1, 4) \}$ in $\mathbb{R}^3$

(e) $\{ (1, -1, 2), (2, 0, 1), (-1, 2, -1) \}$ in $\mathbb{R}^3$

(f) $\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$

(g) $\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$

(h) $\{ x^4 - x^3 + 5x^2 - 8x + 6, -x^4 + x^3 - 5x^2 + 5x - 3, x^4 + 3x^2 - 3x + 5, 2x^4 + 3x^3 + 4x^2 - x + 1, x^3 - x + 2 \}$ in $\mathsf{P}_4(\mathbb{R})$

(i) $\{ x^4 - x^3 + 5x^2 - 8x + 6, -x^4 + x^3 - 5x^2 + 5x - 3, x^4 + 3x^2 - 3x + 5, 2x^4 + x^3 + 4x^2 + 8x \}$ in $\mathsf{P}_4(\mathbb{R})$

Solution: For the sake of brevity, we will not give the detail proofs for the reasoning. Readers are encouraged to work out the details.

(a) Linearly dependent: $2 \cdot \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} + 1 \cdot \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = 0$

(b) Linearly independent

(c) Linearly dependent: $4 \cdot (x^3 - x) - 3 \cdot (2x^2 + 4) + 2 \cdot (-2x^3 + 3x^2 + 2x + 6) = 0$

(d) Linearly dependent: $3 \cdot (1, -1, 2) - 2 \cdot (1, -2, 1) - 1 \cdot (1, 1, 4) = 0$

(e) Linearly independent

(f) Linearly dependent: $3 \cdot \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix} = 0$

(g) Linearly independent

(h) Linearly independent

(i) Linearly dependent: $4 \cdot (x^4 - x^3 + 5x^2 - 8x + 6) + 3 \cdot (-x^4 + x^3 - 5x^2 + 5x - 3) - 3 \cdot (x^4 + 3x^2 - 3x + 5) + 1 \cdot (2x^4 + x^3 + 4x^2 + 8x) = 0$

1.5.8. Let $S = \{ (1, 1, 0), (1, 0, 1), (0, 1, 1) \}$ be a subset of the vector space $\mathbb{F}^3$.

(a) Prove that if $\mathbb{F} = \mathbb{R}$, then S is linearly independent.

(b) Prove that if $\mathbb{F}$ has characteristic 2, then S is linearly dependent.

Solution:

(a) Let $a, b, c \in \mathbb{R}$ such that $a \cdot (1, 1, 0) + b \cdot (1, 0, 1) + c \cdot (0, 1, 1) = 0_{\mathbb{R}^3} = (0, 0, 0)$. Then $(0, 0, 0) = (a + b, a + c, b + c)$ and so $a + b = a + c = b + c = 0$. Thus $2 \cdot (a + b + c) = 0$ and so $a + b + c = 0$. Hence $a = (a + b + c) - (a + b) = 0$, $b = (a + b + c) - (a + c) = 0$, $c = (a + b + c) - (a + b) = 0$. This implies that S is $\mathbb{R}$-linearly independent.

(b) Since $\mathbb{F}$ has characteristic 2, $1 + 1 = 0$. As $1 \neq 0$ and $1 \cdot (1, 1, 0) + 1 \cdot (1, 0, 1) + 1 \cdot (0, 1, 1) = (1+1, 1+1, 1+1) = (0, 0, 0) = 0_{\mathbb{F}^3}$, S is linearly dependent.

1.5.9. Let u and v be distinct vectors in a vector space V. Show that $\{ u, v \}$ is linearly dependent if and only if u or v is a multiple of the other.

Solution:

(a) Suppose $\{ u, v \}$ is linearly dependent. Then there exists scalars a, b not all zero such that $au + bv = 0$. As a, b are not all zero, at least one of them is nonzero. Without loss of generality we may assume that $a \neq 0$. Then $u + \frac{b}{a} \cdot v = 0$ and so $u = -\frac{b}{a} \cdot v$ is a scalar multiple of v.

(b) Suppose u or v is a multiple of the other. Without loss of generality we may assume that u is a multiple of v. Then $u = \lambda v$ for some scalar λ, and so $1 \cdot u - \lambda \cdot v = 0$. As $1 \neq 0$, the scalars are not all zero, and so $\{ u, v \}$ is linearly dependent.

Therefore, $\{ u, v \}$ is linearly dependent if and only if u or v is a scalar multiple of the other.

1.5.16. Prove that a set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.

Solution: Instead of the original proposition, we will prove the following logically equivalent statement:

A set S is linearly dependent if and only if some finite subset of S is

Suppose S is linearly dependent. Then there exists $n \in \mathbb{N}^+$, $v_1, \ldots, v_n \in S$ distinct, and scalars $c_1, \ldots, c_n \in \mathbb{F}$ not all zero such that $\sum_{i=1}^{n} c_i v_i = 0$. This implies that the finite subset $\{ v_1, \ldots, v_n \} \subseteq S$ is linearly dependent.

Suppose S has a finite linearly dependent subset S'. Trivially, $S' \neq \emptyset$. We may then assume that $S' = \{ v_1, \ldots, v_n \}$ with $v_1, \ldots, v_n$ distinct. Then $\sum_{i=1}^{n} c_i v_i = 0$ for some scalars $c_1, \ldots, c_n$ not all zero. As $v_i \in S' \subseteq S$ for all $i \in \{ 1, \ldots, n \}$, this implies that S is also linearly dependent.

Hence, S is linearly dependent if and only if some finite subset of S is. Equivalently, S is linearly independent if and only if every finite subset of S is.

Note

If S itself is a finite set, this proposition does not give us anything new.

1.5.17. Let M be a square upper triangular matrix with nonzero diagonal entries. Prove that the columns of M are linearly independent.

Solution: Let the column vectors be $v_1, \ldots, v_n \in \mathbb{F}^n$ such that $M = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}$. By assumption, $M_{ii} = (v_i)_i \neq 0$ and $M_{ji} = (v_i)_j = 0$ for $i \in \{ 1, \ldots, n \}$, $j \in \{ i+1, \ldots, n \}$.

Let $c_1, \ldots, c_n \in \mathbb{F}$ such that $\sum_{i=1}^{n} c_i v_i = 0$. Then $Mc = 0$ with $c = \begin{pmatrix} c_1 & \ldots & c_n \end{pmatrix}^\mathsf{T} \in \mathbb{F}^n$. Since M is upper triangular, $\det M = \prod_{i=1}^{n} M_{ii} = \prod_{i=1}^{n} (v_i)_i \neq 0$. This implies that M is invertible, and so $c = 0$. Hence $c_1 = \ldots = c_n = 0$. This implies that $v_1, \ldots, v_n$ are linearly independent.

1.5.19. Prove that if $\{ A_1, A_2, \ldots, A_k \}$ is a linearly independent subset of $M_{n \times n}(\mathbb{F})$, then $\{ A_1^\mathsf{T}, A_2^\mathsf{T}, \ldots, A_k^\mathsf{T} \}$ is also linearly independent.

Solution: Let $c_1, \ldots, c_k \in \mathbb{F}$ be such that $\sum_{i=1}^{k} c_i A_i^\mathsf{T} = 0_{n \times n}$. Then $0_{n \times n} = 0_{n \times n}^\mathsf{T} = \left(\sum_{i=1}^{k} c_i A_i^\mathsf{T} \right)^\mathsf{T} = \sum_{i=1}^{k} c_i (A_i^\mathsf{T})^\mathsf{T} = \sum_{i=1}^{k} c_i A_i$. Since $\{ A_1, A_2, \ldots, A_k \}$ is linearly independent, $c_1 = \ldots = c_k = 0$. This implies that $\{ A_1^\mathsf{T}, A_2^\mathsf{T}, \ldots, A_k^\mathsf{T} \}$ is also linearly independent.

1.5.20. Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the functions defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$, where $r \neq s$. Prove that f and g are linearly independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Solution: Let $a, b \in \mathbb{R}$ be such that $af + bg = 0$. Then for all $t \in \mathbb{R}$, $0 = af(t) + bg(t) = ae^{rt} + be^{rt}$. In particular,

$$0 = a + b \quad \text{with } t = 0$$
$$0 = ae^r + be^s \quad \text{with } t = 1$$

Since $r \neq s$, $e^r \neq e^s$. Thus solving the linear system we obtain $a = b = 0$. This implies that f, g are linearly independent.

Note

The proposition can also be shown by noting that the Wronskian $W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} e^{rt} & e^{st} \\ re^{rt} & se^{st} \end{vmatrix} = (s - r)e^{(r+s)t}$ is not identically zero.

1.6.1. Label the following statements as true or false.

(a) The zero vector space has no basis.
(b) Every vector space that is generated by a finite set has a basis.
(c) Every vector space has a finite basis.
(d) A vector space cannot have more than one basis.
(e) If a vector space has a finite basis, then the number of vectors in every basis is the same.
(f) The dimension of $\mathsf{P}_n(\mathbb{F})$ is n.
(g) The dimension of $M_{m\times n}(\mathbb{F})$ is $m+n$.
(h) Suppose that V is a finite-dimensional vector space, that S_1 is a linearly independent subset of V, and that S_2 is a subset of V that generates V. Then S_1 cannot contain more vectors than S_2.
(i) If S generates the vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way.
(j) Every subspace of a finite-dimensional space is finite-dimensional.
(k) If V is a vector space having dimension n, then V has exactly one subspace with dimension 0 and exactly one subspace with dimension n.
(l) If V is a vector space having dimension n, and if S is a subset of V with n vectors, then S is linearly independent if and only if S spans V.

Solution:

(a) False

(b) True

(c) False

(d) False

(e) True

(f) False. It is $n+1$

(g) False. It is $m\cdot n$

(h) True

(i) False. This only holds if S is linearly independent

(j) True

(k) True

(l) True

1.6.3. Determine which of the following sets are bases for $\mathsf{P}_2(\mathbb{R})$.

(a) $\{\, -1-x+2x^2, 2+x-2x^2, 1-2x+4x^2 \,\}$
(b) $\{\, 1+2x+x^2, 3+x^2, x+x^2 \,\}$
(c) $\{\, 1-2x-2x^2, -2+3x-x^2, 1-x+6x^2 \,\}$
(d) $\{\, -1+2x+4x^2, 3-4x-10x^2, -2-5x-6x^2 \,\}$
(e) $\{\, 1+2x-x^2, 4-2x+x^2, -1+18x-9x^2 \,\}$

Solution: For the sake of brevity, we will not give the detail proofs for the reasoning. Readers are encouraged to work out the details.

(a) The set does not form a basis as it is linearly dependent: $5\cdot(-1-x+2x^2)+3\cdot(2+x-2x^2)-1\cdot(1-2x+4x^2)=0$

(b) The set forms a basis

(c) The set forms a basis

(d) The set forms a basis

(e) The set does not form a basis as it is linearly dependent: $7\cdot(1+2x-x^2)-2\cdot(4-2x+x^2)-1\cdot(-1+18x-9x^2)=0$

1.6.22. Let W_1 and W_2 be subspaces of a finite-dimensional vector space V. Determine necessary and sufficient conditions on W_1 and W_2 so that $\dim(W_1 \cap W_2) = \dim(W_1)$.

Solution: The sufficient and necessary condition is that $W_1 \subseteq W_2$.

Suppose $W_1 \subseteq W_2$. Then $W_1 \cap W_2 = W_1$ and so $\dim(W_1 \cap W_2) = \dim(W_1)$.

Suppose $\dim(W_1 \cap W_2) = \dim(W_1)$. Since $W_1 \cap W_2 \subseteq W_1$ and that W_1 is finite-dimensional, $W_1 \cap W_2 = W_1$. By set theory, this implies that $W_1 \subseteq W_2$.

Thus $\dim(W_1 \cap W_2) = \dim(W_1)$ if and only if $W_1 \subseteq W_2$.

Note

This does not hold if V is not finite-dimensional.

1.6.28. Let V be a finite-dimensional vector space over $\mathbb{C}$ with dimension n. Prove that if V is now regarded as a vector space over $\mathbb{R}$, then $\dim V = 2n$.

Solution: Denote the corresponding real vector space as $V_{\mathbb{R}}$. Note that V and $V_{\mathbb{R}}$ share the same underlying set, which we denote as S.

Let $\beta \subseteq S$ be a $\mathbb{C}$-basis of V. By assumption, we may assume that $\beta = \{ v_1, \dots, v_n \}$.

Since β is a $\mathbb{C}$-basis of V, $v_1, \dots, v_n, i \cdot v_1, \dots, i \cdot v_n$ are distinct. Let $\gamma = \{ v_1, \dots, v_n, i \cdot v_1, \dots, i \cdot v_n \} \subseteq S$. It suffices to show that γ is a basis of $V_{\mathbb{R}}$ as $|\gamma| = 2n$.

Let $v \in S$. Since β is a $\mathbb{C}$-basis of V, there exists $c_1, \dots, c_n \in \mathbb{C}$ such that $v = \sum_{k=1}^{n} c_k v_k = \sum_{k=1}^{n} (a_k + i b_k) v_k = \sum_{k=1}^{n} a_k v_k + \sum_{k=1}^{n} b_k (i \cdot v_k) \in \mathrm{Span}_{\mathbb{R}}(\gamma)$ where $a_k = \Re c_k, b_k = \Im c_k \in \mathbb{R}$ are the real part and the imaginary part of c_k respectively. As $\gamma \subseteq S$, this implies that γ $\mathbb{R}$-spans $V_{\mathbb{R}}$.

Let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$ such that $0 = \sum_{k=0}^{n} a_k v_k + \sum_{k=0}^{n} b_k (i \cdot v_k)$. Then $0 = \sum_{k=0}^{n} (a_k + i b_k) v_k$ where $a_k + i b_k \in \mathbb{C}$ for all k. As β is a $\mathbb{C}$-basis of V, $a_k + i b_k = 0$ for all k, thus $a_k = b_k = 0$ for all k. This implies that γ is $\mathbb{R}$-linearly independent.

Therefore γ is a $\mathbb{R}$-basis of $V_{\mathbb{R}}$, and so $\dim_{\mathbb{R}}(V_{\mathbb{R}}) = 2n$.

1.6.31. Let W_1 and W_2 be subspaces of a vector space V having dimensions m and n, respectively, where $m > n$.

(a) Prove that $\dim(W_1 \cap W_2) \leq n$.

(b) Prove that $\dim(W_1 + W_2) \leq m + n$.

Solution:

(a) As $W_1 \cap W_2 \subseteq W_2$, $\dim(W_1 \cap W_2) \leq \dim(W_2) = n$

(b) By Question 1.6.29(a), $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \leq \dim(W_1) + \dim(W_2) = m + n$

1.6.32. (a) Find an example of subspaces W_1 and W_2 of $\mathbb{R}^3$ with dimensions m and n, where $m > n > 0$, such that $\dim(W_1 \cap W_2) = n$.

(b) Find an example of subspaces W_1 and W_2 of $\mathbb{R}^3$ with dimensions m and n, where $m > n > 0$, such that $\dim(W_1 + W_2) = m + n$.

Solution:

(a) Let $W_1 = \{ (x, y, 0) : x, y \in \mathbb{R} \}$, $W_2 = \{ (x, 0, 0) : x \in \mathbb{R} \}$. It is easy to see that W_1, W_2 are subspaces of $\mathbb{R}^3$, $W_1 \supseteq W_2$, $\dim(W_1) = 2$, $\dim(W_2) = 1 = \dim(W_1 \cap W_2)$

(b) Let $W_1 = \{ (x, y, 0) : x, y \in \mathbb{R} \}$, $W_2 = \{ (0, 0, z) : z \in \mathbb{R} \}$. It is easy to see that W_1, W_2 are subspaces of $\mathbb{R}^3$, $W_1 + W_2 = \mathbb{R}^3$, $\dim(W_1) = 2$, $\dim(W_2) = 1$, $\dim(W_1 + W_2) = 3 = \dim(W_1) + \dim(W_2)$

1.6.33. (a) Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$. If β_1 and β_2 are bases for W_1 and W_2, respectively, show that $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cup \beta_2$ is a basis for V.

(b) Conversely, let β_1 and β_2 be disjoint bases for subspaces W_1 and W_2, respectively, of a vector space V. Prove that if $\beta_1 \cup \beta_2$ is a basis for V, then $V = W_1 \oplus W_2$.

Solution:

(a) Suppose $\beta_1 \cap \beta_2 \neq \emptyset$. Then there exists $v \in \beta_1 \cap \beta_2 \subset W_1 \cap W_2 = \{0\}$. Then $0 \in \beta_1 \cap \beta_2$. As β_1 and β_2 are basis, they are linearly independent. Contradiction arises. Hence $\beta_1 \cap \beta_2 = \emptyset$.

Trivially, $\text{Span}(\ \beta_1 \cup \beta_2\ ) = \text{Span}(\ \beta_1\ ) + \text{Span}(\ \beta_2\ ) = W_1 + W_2 = V$. It then remains to show that $\beta_1 \cup \beta_2$ is linearly independent.

Let $n, m \in \mathbb{N}$, $c_1, \ldots, c_n, d_1, \ldots, d_m \in \mathbb{F}$, and $v_1, \ldots, v_n \in \beta_1$, $w_1, \ldots, w_n \in \beta_2$ be distinct such that $\sum_{i=1}^n c_i v_i + \sum_{i=1}^m d_i w_i = 0$. Then $\sum_{i=1}^n c_i v_i = -\sum_{i=1}^m d_i w_i \in \text{Span}(\ \beta_1\ ) \cap \text{Span}(\ \beta_2\ ) = W_1 \cap W_2 = \{0\}$. This implies that $\sum_{i=1}^n c_i v_i = \sum_{i=1}^m d_i w_i = 0$. As β_1, β_2 are linearly independent, $c_1 = \ldots = c_n = d_1 = \ldots = d_m = 0$. As $n, m, v_1, \ldots, v_n, w_1, \ldots, w_m$ are arbitrary, this implies that $\beta_1 \cap \beta_2$ is linearly independent.

Hence $\beta_1 \cup \beta_2$ is a basis of V.

(b) As $\beta_1 \cup \beta_2$ is a basis of V, $V = \text{Span}(\ \beta_1 \cup \beta_2\ ) = \text{Span}(\ \beta_1\ ) + \text{Span}(\ \beta_2\ ) = W_1 + W_2$.

Trivially, $W_1 \cap W_2 \supseteq \{0\}$. Let $v \in W_1 \cap W_2$. Then $v \in W_1 = \text{Span}(\ \beta_1\ )$ and $v \in W_2 = \text{Span}(\ \beta_2\ )$. Let $n, m \in \mathbb{N}$, $v_1, \ldots, v_n \in \beta_1$, $w_1, \ldots, w_m \in \beta_2$ be distinct such that $v = \sum_{i=1}^n c_i v_i = \sum_{i=1}^m d_i w_i$ and so $\sum_{i=1}^n c_i v_i - \sum_{i=1}^m d_i w_i = 0$. As $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cup \beta_2$ is linearly independent, $c_1 = \ldots = c_n = d_1 = \ldots = d_m = 0$, and so $v = \sum_{i=1}^n c_i v_i = 0$. As v is arbitrary, $W_1 \cap W_2 = \{0\}$.

This implies that $V = W_1 \oplus W_2$.

Note

See also Question 1.4.15 in the previous homework.

1.6.34. (a) Prove that if W_1 is any subspace of a finite-dimensional vector space V, then there exists a subspace W_2 of V such that $V = W_1 \oplus W_2$.

(b) Let $V = \mathbb{R}^2$ and $W_1 = \{\ (a_1, 0) : a_1 \in \mathbb{R}\ \}$. Give examples of two different subspaces W_2 and W_2' such that $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_2'$.

Solution:

(a) Since V is finite-dimensional, so is W_1. Let β be a basis of W_1. By Extension Theorem, we may extend β to a basis $\gamma = \beta \cup \alpha$ of V with $\alpha \subseteq V$ and $\beta \cap \alpha = \emptyset$. As γ is a basis, α is linearly independent. Let $W_2 = \text{Span}(\ \alpha\ )$. Then α is a basis of W_2. By Question 1.6.33(a), $V = W_1 \oplus W_2$.

(b) Let $W_2 = \{\ (x, x) : x \in \mathbb{R}\ \}$ and $W_2' = \{\ (x, -x) : x \in \mathbb{R}\ \}$. It is easy to see that W_2, W_2' are distinct subspaces of $V = \mathbb{R}^2$, $W_1 \cap W_2 = W_1 \cap W_2' = \{0\}$, and $W_1 + W_2 = W_1 + W_2' = V$, so $V = W_1 \oplus W_2 = W_1 \oplus W_2'$.

2.1.1. Label the following statements as true or false. In each part, V and W are finite-dimensional vector spaces (over $\mathbb{F}$), and T is a function from V to W.

(a) If T is linear, then T preserves sums and scalar products.

(b) If $T(x + y) = T(x) + T(y)$, then T is linear.

(c) T is one-to-one if and only if the only vector x such that $T(x) = 0$ is $x = 0$.

(d) If T is linear, then $T(0_V) = 0_W$.

(e) If T is linear, then $\text{nullity}(T) + \text{rank}(T) = \dim(W)$.

(f) If T is linear, then T carries linearly independent subsets of V onto linearly independent subsets of W.

(g) If $T, U : V \to W$ are both linear and agree on a basis for V, then $T = U$.

(h) Given $x_1, x_2 \in V$ and $y_1, y_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$.

Solution:

(a) True

(b) False. See Question 2.1.38.

(c) False. This only holds if T is linear.

(d) True

(e) False. This only holds if $\dim(V) = \dim(W)$.

(f) True. See Question 2.1.14.

(g) True

(h) False. Consider the case where $x_1 = x_2$ and $y_1 \neq y_2$.

2.1.9. In this exercise, $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a function. For each of the following parts, state why T is not linear.

(a) $T(a_1, a_2) = (1, a_2)$

(b) $T(a_1, a_2) = (a_1, a_1^2)$

(c) $T(a_1, a_2) = (\sin a_1, 0)$

(d) $T(a_1, a_2) = (|a_1|, a_2)$

(e) $T(a_1, a_2) = (a_1 + 1, a_2)$

Solution: For the sake of brevity, we will only give brief explanations on why the mappings are not linear rather than complete proofs.

(a) $T(0, 0) = (1, 0) \neq (0, 0)$

(b) $2 \cdot T(1, 0) = (2, 2) \neq (2, 4) = T(2 \cdot (1, 1))$

(c) $2 \cdot T(\pi/2, 0) = (2, 0) \neq (-1, 0) = T(2 \cdot (\pi/2, 0))$

(d) $-1 \cdot T(1, 0) = (-1, 0) \neq (1, 0) = T(-1 \cdot (1, 0))$

(e) $T(0, 0) = (1, 0) \neq (0, 0)$

2.1.15. Define

$$T : \mathsf{P}(\mathbb{R}) \to \mathsf{P}(\mathbb{R}) \quad \text{by} \quad T(f) = \int_0^x f(t)\,\mathrm{d}t$$

Prove that T linear and one-to-one, but not onto.

Solution: Let $f, g \in \mathsf{P}(\mathbb{R})$, $c \in \mathbb{R}$. Then

- $T(f + g) = \int_0^x f(t) + g(t)\,\mathrm{d}t = \int_0^x f(t)\,\mathrm{d}t + \int_0^x g(t)\,\mathrm{d}t = T(f) + T(g)$
- $T(cf) = \int_0^x cf(t)\,\mathrm{d}t = c\int_0^x f(t)\,\mathrm{d}t = cT(f)$

Thus T is linear.

Let $f \in \mathsf{N}(T)$. WLOG we may assume that $f(x) = \sum_{i=0}^n c_i x^i$ for some $n \in \mathbb{N}$, $c_0, \ldots, c_n \in \mathbb{R}$. Then $0 = T(f) = \int_0^x f(t)\,\mathrm{d}t = \sum_{i=0}^n \int_0^x c_i t^i\,\mathrm{d}t = \sum_{i=0}^n \frac{c_i}{i+1} x^{i+1}$. By comparing coefficients, $\frac{c_i}{i+1} = 0$ for all $i \in \{0, \ldots, n\}$, thus $c_0 = \ldots = c_n = 0$, $f = 0$. This implies that T is one-to-one.

Suppose there exists $f \in \mathsf{P}(\mathbb{R})$ such that $T(f) = \mathbf{1} \in \mathsf{P}(\mathbb{R})$ is the constant 1 polynomial. Then for all $a \in \mathbb{R}$, $1 = T(f)(a) = \int_0^a f(t)\,\mathrm{d}t$. In particular, $1 = T(f)(0) = \int_0^0 f(t)\,\mathrm{d}t = 0$. Contradiction arises. Hence $1 \notin \mathsf{R}(T)$. This implies that T is not onto.

Note

You can also show the last part by comparing the coefficients of $T(f)$ with $\mathbf{1}$.

2.1.19. Give an example of distinct linear transformations T and U such that $\mathsf{N}(T) = \mathsf{N}(U)$ and $\mathsf{R}(T) = \mathsf{R}(U)$.

Solution: Consider $V = \mathbb{R}$ being the (usual) real numbers, $T, U : V \to V$ be defined by $T(x) = x$ and $U(x) = -x$ for all $x \in V = \mathbb{R}$. Then it is easy to see that T, U are distinct, linear, and $\mathsf{N}(T) = \{0\} = \mathsf{N}(U)$, $\mathsf{R}(T) = V = \mathsf{R}(U)$.

2.1.26. Let V be a vector space and W_1 and W_2 be subspaces of V such that $V = W_1 \oplus W_2$. Assume that $T : V \to V$ is the projection on W_1 along W_2.

(a) Prove that T is linear and $W_1 = \{ x \in V : T(x) = x \}$.

(b) Prove that $W_1 = \mathsf{R}(T)$ and $W_2 = \mathsf{N}(T)$.

(c) Describe T if $W_1 = V$.

(d) Describe T if W_1 is the zero subspace.

Solution:

(a) We first show that T is linear.

Let $v, v' \in V$ and $c \in \mathbb{F}$. As $V = W_1 \oplus W_2$, there exists $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$ such that $v = w_1 + w_2$, $v' = w_1' + w_2'$. By definition, $T(v) = w_1$ and $T(v') = w_1'$. Then

- $T(v + v') = T(w_1 + w_1' + w_2 + w_2') = w_1 + w_1' = T(v) + T(v')$ as $w_1 + w_1' \in W_1$ and $w_2 + w_2' \in W_2$.
- $T(cv) = T(cw_1 + cw_1') = cw_1 = cT(v))$ as $cw_1 \in W_1$.

Since v, v', c are arbitrary, T must be linear.

Let $S = \{ x \in V : T(x) = x \}$.

Let $w \in W_1$. Then $w = w_1 + 0$ with $w_1 \in W_1$ and $0 \in W_2$. As T is a projection, we must have $T(w) = w$. As w is arbitrary, $W_1 \subseteq S$.

Let $x \in S \subseteq V$. Then $T(x) = x$. As $x \in V = W_1 \oplus W_2$, there exists $w_1 \in W_1$, $w_2 \in W_2$ such that $x = w_1 + w_2$. As T is a projection, $x = T(x) = w_1 \in W_1$. As x is arbitrary, $S \subseteq W_1$.

Therefore, $W_1 = \{ x \in V : T(x) = x \}$.

(b) By the previous part, $W_1 = \{ x \in V : T(x) = x \} = \{ T(x) \in V : x \in V, T(x) = x \} \subseteq \mathsf{R}(T)$.

Let $v \in \mathsf{R}(T)$. Then there exists $x \in V$ such that $v = T(x)$. As $x \in V = W_1 \oplus W_2$, there exists $w_1 \in W_1$, $w_2 \in W_2$ such that $x = w_1 + w_2$. As T is a projection, $v = T(x) = w_1 \in W_1$. As v is arbitrary, $\mathsf{R}(T) \subseteq W_1$.

Hence $W_1 = \mathsf{R}(T)$.

Let $w \in W_2$. Then $w = 0 + w$ with $0 \in W_1$ and $w \in W_2$. As T is a projection, $T(w) = 0$, so $w \in \mathsf{N}(T)$. As w is arbitrary, $W_2 \subseteq \mathsf{N}(T)$.

Let $x \in \mathsf{N}(T)$. Then $T(x) = 0$. Since $x \in V$. there exists $w_1 \in W_1$, $w_2 \in W_2$ such that $x = w_1 + w_2$. As T is a projection, $0 = T(x) = w_1$ and so $x = w_2 \in W_2$. As x is arbitrary, $\mathsf{N}(T) \subseteq W_2$.

Hence $W_2 = \mathsf{N}(T)$.

(c) Suppose $W_1 = V$. Then by part (a), $V = \{ x \in V : T(x) = x \}$, so $T(x) = x$ for all $x \in V$. This implies that T is the identity map on V.

(d) Suppose $W_1 = \{0\}$. Then $W_2 = W_2 + \{0\} = W_2 + W_1 = V$. So by part (b), $V = W_2 = \mathsf{N}(T)$. This implies that T is the zero map.

Note

In this question, we do not need to verify that T is well-defined (unlike the next question, Question 2.1.27). This is because we are already given a mapping T and assume that it has a certain property (being a projection).

2.1.27. Suppose that W is a subspace of a finite-dimensional vector space V.

(a) Prove that there exists a subspace W' and a function $T : V \to V$ such that T is a projection on W along W'.

(b) Give an example of a subspace W of a vector space V such that there are two projections on W along two (distinct) subspaces.

Solution:

(a) Since V is finite-dimensional, W is also finite-dimensional. Let $\beta \subseteq W$ be a basis of W. By Extension Theorem, β can be extended to a basis $\alpha = \beta \cup \gamma$ of V where $\gamma \subseteq V$ and $\gamma \cap \beta = \emptyset$.

Let $W' = \text{Span}(\gamma) \subseteq V$. Then W' is a subspace of V. Also, $V = \text{Span}(\alpha) = \text{Span}(\beta \cup \gamma) = \text{Span}(\beta) + \text{Span}(\gamma) = W_1 + W_2$. Furthermore, $\dim(W_1) + \dim(W_2) = | \beta | + | \gamma | = | \beta | = \dim(V)$, so by Question 1.6.29, $V = W_1 \oplus W_2$.

By Question 1.3.30 (in the previous homework), for each $x \in V$ there exist unique $w_1 \in W_1$ and $w_2 \in W_2$ such that $x = w_1 + w_2$. Define $T : V \to V$ by $T(x) = w_1$ for each $x \in V$ where w_1 is from the decomposition. Then T is well-defined from the existence and uniqueness of such decomposition.

It remains to show that T is a projection on W along W'. Let $x \in V$ be such that $x = w_1 + w_2$ for some $w_1 \in W_1$, $w_2 \in W_2$. Since $V = W_1 \oplus W_2$, such decomposition is unique. So by the definition of T we have $T(x) = w_1$. As x is arbitrary, T is a projection on W_1 along W_2.

Hence there exists a subspace W' and a function $T : V \to V$ such that T is a projection on W along W'.

(b) Let $V = \mathbb{R}^2$ be the (usual) real plane, $W_1 = \{ (x, 0) \in V : x \in \mathbb{R} \}$, $W_2 = \{ (x, x) \in V : x \in \mathbb{R} \}$, $W_2' = \{ (x, -x) \in V : x \in \mathbb{R} \}$. It is easy to see that W_1, W_2, W_2' are subspaces of V, $W_2 \neq W_2'$, and $V = W_1 \oplus W_2 = W_1 \oplus W_2'$. Define T, T' to be the projection on W_1 along W_2 and the projection on W_1 along W_2' respectively as in the last part.

By Question 2.1.26 part (b), we have $\mathsf{N}(T) = W_2 \neq W_2' = \mathsf{N}(T')$, so $T \neq T'$.

2.1.28. Assume that $T : V \to V$ is linear. Prove that $\{0\}, V, \mathsf{R}(T), \mathsf{N}(T)$ are all T-invariant.

Solution:

- $T\{0\} = \{T(0)\} = \{0\}$
- By definition, for each $v \in V$, $T(v) \in V$, thus $T(V) \subseteq V$
- For each $v \in \mathsf{R}(T)$ we have by definition of range that $T(v) \in \mathsf{R}(T)$, so $T(\mathsf{R}(T)) \subseteq \mathsf{R}(T)$
- For each $v \in \mathsf{N}(T)$ we have $T(v) = 0 \in \mathsf{N}(T)$, so $T(\mathsf{N}(T)) \subseteq \mathsf{N}(T)$

Therefore, $\{0\}, V, \mathsf{R}(T), \mathsf{N}(T)$ are all T-invariant.

2.1.31. Assume that W is a subspace of a vector space V and that $T : V \to V$ is linear. Suppose that $V = \mathsf{R}(T) \oplus W$ and W is T-invariant.

(a) Prove that $W \subseteq \mathsf{N}(T)$.

(b) Show that if V is finite-dimensional, then $W = \mathsf{N}(T)$.

(c) Show by example that the conclusion of (b) is not necessarily true if V is not finite-dimensional.

Solution:

(a) Let $w \in W$. Since W is T-invariant, $T(w) \subseteq W$. By definition, $T(w) \in \mathsf{R}(T)$, so $T(w) \in W \cap \mathsf{R}(T)$. As $V = \mathsf{R}(T) \oplus W$, $\mathsf{R}(T) \cap W = \{0\}$. Hence $T(w) = 0$, $w \in \mathsf{N}(T)$. As w is arbitrary, $W \subseteq \mathsf{N}(T)$.

(b) By dimension theorem and the result of Question 1.6.29(b), $\dim(V) = \dim(\mathsf{R}(T)) + \dim(\mathsf{N}(T)) \geq \dim(\mathsf{R}(T)) + \dim(W) = \dim(V)$. Hence $\dim(\mathsf{R}(T)) + \dim(\mathsf{N}(T)) = \dim(\mathsf{R}(T)) + \dim(W)$ and so $\dim(\mathsf{N}(T)) = \dim(W)$. As $W \subseteq \mathsf{N}(T)$, $W = \mathsf{N}(T)$.

(c) Let V be the (real) vector space of real sequences. Let $T : V \to V$ be the left shift operator from Question 2.1.28. Then T is linear, and $\mathsf{R}(T) = V$. Let $W = \{0\} \subseteq V$. Then W is T-invariant, and $V = V \oplus \{0\} = \mathsf{R}(T) \oplus W$. However, $\mathsf{N}(T) = \{ (0, a_2, a_3, \dots) \in V : a_2, a_3, \dots \in \mathbb{R} \} \neq \{0\} = W$.

2.1.32. Assume that W is a subspace of a vector space V and that $T : V \to V$ is linear. Suppose that W is T-invariant. Prove that $\mathsf{N}(T_W) = \mathsf{N}(T) \cap W$.

Solution:

(a) Let $w \in \mathsf{N}(T_W)$. By definition, $w \in W$. Also, $T(w) = T_W(w) = 0$, so $w \in \mathsf{N}(T)$. This implies that $w \in W \cap \mathsf{N}(T)$. As w is arbitrary, $\mathsf{N}(T_W) \subseteq W \cap \mathsf{N}(T)$.

(b) Let $w \in W \cap \mathsf{N}(T)$. Then $T(w) = 0$. As $w \in W$, $T_W(w)$ is well-defined and $T_W(w) = T(w) = 0$, so $w \in \mathsf{N}(T_W)$.

Therefore $\mathsf{N}(T_W) = W \cap \mathsf{N}(T)$.

2.1.39. Prove that there is an additive function $T : \mathbb{R} \to \mathbb{R}$ that is not linear.

Solution: Let β be a $\mathbb{Q}$-basis of $\mathbb{R}$. By cardinality argument, β is infinite, and so has at least two distinct elements. Let $x, y \in \beta$ be distinct. Define $T : \mathbb{R} \to \mathbb{R}$ by

$$T(v) = \begin{cases} y & \text{if } v = x \\ x & \text{if } v = y \\ v & \text{if } v \notin \{x, y\} \end{cases}$$

for $v \in \beta$ and extend $\mathbb{Q}$-linearly. As β is a $\mathbb{Q}$-basis, T is a well-defined $\mathbb{Q}$-linear map. In particular, T is additive.

We now show that T is not $\mathbb{R}$-linear. As $x, y \in \beta$, neither of them is 0. Let $c = y/x \in \mathbb{R}$. Then $T(cx) = T(y) = x$, and $cT(x) = cy = y^2/x$.

If T is $\mathbb{R}$-linear, we would have $x = T(cx) = cT(x) = y^2/x$. This would imply that $y = x$ or $y = -x$ and so $\beta \supseteq \{x, y\}$ is not $\mathbb{Q}$-linearly independent, which is a contradiction to the assumption that β is a $\mathbb{Q}$-basis. Hence T is not $\mathbb{R}$-linear.

In particular, there exists an additive function from $\mathbb{R}$ to $\mathbb{R}$ that is not linear.

Note

Compare this with Question 2.1.37 and 2.1.38.

The explicit construction of β turns out to be a complicated matter. See this (and other relevant) answer on MO.

With β we can show a few more "surprising" results, including: for every positive integer $n \geq 2$, there exists n periodic functions $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ such that $x = f_1(x) + \ldots + f_n(x)$ for all $x \in \mathbb{R}$.

2.1.40. Let V be a vector space and W be a subspace of V. Define the mapping $\eta : V \to V/W$ by $\eta(v) = v + W$ for $v \in V$.

(a) Prove that η is a linear transformation from V onto V/W and that $\mathsf{N}(\eta) = W$.

(b) Suppose that V is finite-dimensional. Use (a) and the dimension theorem to derive a formula relating $\dim(V)$, $\dim(W)$, and $\dim(V/W)$.

(c) Compare the method of solving (b) with the method of deriving the same result as outlined in Exercise 35 of Section 1.6.

Solution:

(a) We first show that η is linear. Let $x, y \in V$, $c \in \mathbb{F}$. Then

- $\eta(x + y) = (x + y) + W = (x + W) + (y + W) = \eta(x) + \eta(y)$
- $\eta(cx) = (cx) + W = c(x + W) = c\eta(x)$

As x, y, c are arbitrary, η is linear.

For each $S \in V/W$, we have by definition of V/W that $S = v + W = \eta(v)$ for some $v \in V$. Thus η is onto.

By the result of Question 1.3.21 (from the previous homework), for each $v \in V$, $\eta(v) = v + W = W = 0_{V/W}$ if and only if $v \in W$. This implies that $\mathsf{N}(\eta) = W$.

(b) By dimension theorem, $\dim(V) = \text{nullity}(\eta) + \text{rank}(\eta) = \dim(\mathsf{N}(\eta)) + \dim(\mathsf{R}(\eta)) = \dim(W) + \dim(V/W)$.

(c) Exercise 3.5 of Section 1.6 is done by constructing a basis for V/W directly from extending a basis of W to a basis of V. In view of the dimension theorem employed here, the proofs are similar: dimension theorem is also shown by extending a basis of the null space and arguing that the extended part corresponds to the range; in Exercise 3.5, the basis constructed is exactly the extended part.