Week 3:

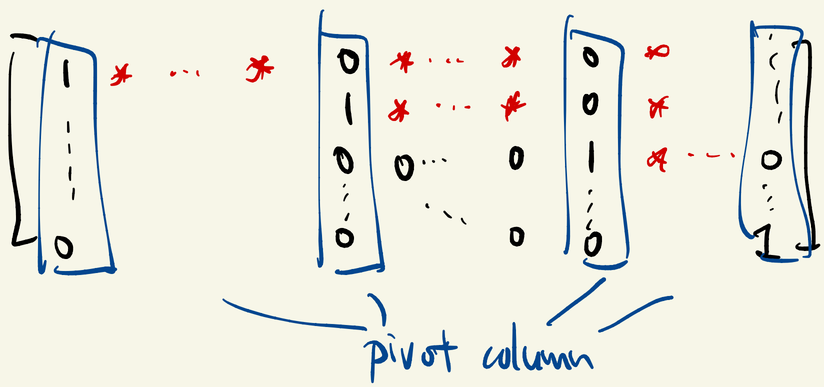
Some remarks about solving system of linear eq. using RREF:

$$(*)\quad \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$\rightsquigarrow$ Consider augmented matrix $[A|b]$.

$\rightsquigarrow$ turn $[A|b]$ into RREF by Gaussian elimination

i.e. $[A|b] \xrightarrow{\text{row operation}}$



Thm: The system of linear equation of n unknowns $(*)$ consistent (solvable) iff the $(n+1)$-th column is Not pivot.
iff the last non-zero row $\neq (0, \cdots 0, 1)$.

Further remark:

Defn: Given a matrix, the rank of the matrix A is defined to be the no. of non-zero row in its RREF.

Thm: If the system of linear equation with n unknowns is consistent, then $\operatorname{rank}[A|b] \leq n$.

If rank $= n$, then the solution is unique
otherwise there are ∞ many solution.

Eg: $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ 1 & 1 & 0 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\therefore$ rank $= 2$ < No. of variable $= 3$

$\Rightarrow \infty$ many sol.

New topics: Matrix.

Algebra:

(of $p\times q$ type)

- two matrix are identical $A=B$, if $A_{ij} = B_{ij}\ \forall i, j$.

Addition of matrix:

Given two $p\times q$ matrix A, B,

$A+B = p\times q$ matrix given by

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

Eg: $A = \begin{bmatrix} a_{11} & \cdots & a_{1q} \\ a_{p1} & \cdots & a_{pq} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & & \vdots \\ b_{p1} & & b_{pq} \end{bmatrix}$

then $A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1q}+b_{1q} \\ & \ddots & & \vdots \\ a_{p1}+b_{p1} & & & a_{pq}+b_{pq} \end{bmatrix}$

Scalar multiplication: If $\lambda \in \mathbb{R}$, then $\lambda A = p\times q$ matrix s.t.

$(\lambda A)_{ij} = \lambda A_{ij}$ i.e. $\lambda \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{pq} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots \\ & \ddots & \\ & & \lambda a_{pq} \end{bmatrix}$

additive inverse $-A$ (s.t. $-A+A=0=$ zero matrix)

$$(-A)_{ij} = -A_{ij} \quad \text{i.e.} \quad -\begin{pmatrix} a_{11} & & \\ & \ddots & \\ a_{p1} & & a_{pq} \end{pmatrix} = \begin{pmatrix} -a_{11} & & \\ & \ddots & \\ -a_{p1} & & -a_{pq} \end{pmatrix}$$

properties:

① $(A+B)+C = A+(B+C)$

② $A+B = B+A$

③ $A+(-A) = 0 = (-A)+A$.

④ $A+0 = A = 0+A$.

⑤ $\lambda(AB) = (\lambda A)B$

⑥ $(\lambda+\tilde{\lambda})A = \lambda A + \tilde{\lambda} A$

⑦ $\lambda(A+B) = \lambda A + \lambda B$

⑧ $1 \cdot C = C$.

why care??

$A+B$: represent addition of systems

λA : scalar multiplication of system.

More Motivation:

$$\star \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$(t_1, t_2, \ldots, t_n) =$ solution to $\star$ means:

$$\begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{pq} \end{bmatrix} \xrightarrow{\text{"applied"}} \underset{\text{"action"}}{[t_1, \ldots, t_n]} \overset{\text{yields}}{\rightsquigarrow} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

"Action" $\rightsquigarrow$ "product of matrix."

Need a operation s.t. $A \cdot \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

Define: for $A = p \times q$ matrix

$B = q \times m$ matrix

$AB = p \times m$ matrix s.t. $(AB)_{ij} = \sum_{l=1}^{q} A_{il} B_{lj}$

i.e. $\begin{pmatrix} a_{11} & \cdots & a_{1q} \\ \vdots & & \\ a_{p1} & \cdots & a_{pq} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \\ b_{q1} & \cdots & b_{qm} \end{pmatrix}$

$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1q}b_{q1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1q}b_{q2} & \cdots \\ & \ddots & \end{pmatrix}$

Hence. $\begin{cases} a_{11}t_1 + \cdots + a_{1n}t_n = b_1 \\ \vdots \\ a_{m1}t_1 + \cdots + a_{mn}t_n = b_m \end{cases}$ is equivalent to

$\Updownarrow$

$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & & a_{mn} \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \end{pmatrix} = 2 \times 5 \text{ matrix}$$

$$B = \begin{pmatrix} 0 & 5 \\ 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{pmatrix} = 5 \times 2 \text{ matrix}$$

$$\Rightarrow \quad AB = 2 \times 2 \text{ matrix}$$

$$= \begin{pmatrix} 1\cdot 0 + 2\cdot 1 + 3\cdot 2 + 4\cdot 3 + 5\cdot 4 & 1\cdot 5 + 2\cdot 6 + 3\cdot 7 + 4\cdot 8 + 5\cdot 9 \\ 2\cdot 0 + 3\cdot 1 + 4\cdot 2 + 5\cdot 3 + 6\cdot 4 & 2\cdot 5 + 3\cdot 6 + 4\cdot 7 + 5\cdot 8 + 6\cdot 9 \end{pmatrix}$$

$$= \begin{pmatrix} 40 & 115 \\ 50 & 150 \end{pmatrix} \quad \#$$

"Identity" element: $I_p = p \times p$ matrix $= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$

$$(I_p)_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

then $\begin{cases} I_p \underset{\uparrow}{A} = A & \quad (A: p \times q \text{ matrix}) \\ A I_q = A \end{cases}$

since

$$(I_p A)_{ij} = \sum_{l=1}^{p} (I_p)_{il} A_{lj} = A_{ij}$$

$$(A I_q)_{ij} = \sum_{l=1}^{q} A_{il} (I_q)_{lj} = A_{ij} \quad \#$$

Thm: ① $\lambda(AB) = (\lambda A)B = A(\lambda B)$

② $A(B+C) = AB + AC$

③ $(A+B)C = AC + BC$

$\boxed{\text{Rmk: } AB \neq BA \text{ in general}}$

pf:

①: $\lambda(AB)_{ij} = \sum_{l=1}^{q} \lambda A_{il} B_{lj}$

②: $(A(B+C))_{ij}$

$$= \sum_{l=1}^{q} A_{il}(B+C)_{lj}$$

$$= \sum_{l=1}^{q} A_{il}B_{lj} + \sum_{l=1}^{q} A_{il}C_{lj}$$

③: similar.

Thm: A: $m\times n$ matrix

B: $n \times p$ matrix

C: $p \times l$ matrix

then $(AB)C = A(BC) = m \times l$ matrix

pf: suffices to show that $\forall i = 1, 2, \dots, m;\ j = 1, 2, \dots, l$

$$((AB)C)_{ij} = (A(BC))_{ij}$$

$$\text{L.H.S} = ((AB)C)_{ij} = \sum_{q=1}^{p} (AB)_{iq} C_{qj}$$

$$= \sum_{q=1}^{l} \left(\sum_{k=1}^{n} A_{ik} B_{kq} \right) C_{qj}$$

$$= \sum_{q=1}^{l} \sum_{k=1}^{n} A_{ik} B_{kq} C_{qj} = \sum_{k=1}^{n} A_{ik} \left(\sum_{q=1}^{l} B_{kq} C_{qj} \right)$$

$$= \sum_{k=1}^{n} A_{ik} (BC)_{kj} = (A(BC))_{ij} = \text{R.H.S} \quad \#$$

Recall: we may read (S) $\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$

as $\underbrace{Ax = b}$ where $A = \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{mn} \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$.

matrix representation of (S) $LS(A, b)$

Alternatively: $A = \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{array}\right] = [A_1 \; A_2 \cdots A_n]$

where each $A_i = m \times 1$ matrix $=$ i-th column of A

then $Ax = b \iff \sum_{i=1}^{n} x_i A_i = b$ ← $m\times 1$ matrix

scalar $\quad$ $m \times 1$ matrix

$\Updownarrow$

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Call: vector representation of the system (S).

Example: (S): $\begin{cases} x_1 + 2x_2 \qquad\quad + x_4 = 7 \\ x_1 + x_2 + x_3 - x_4 = 3 \\ 3x_1 + x_2 + 5x_3 - 7x_4 = 1 \end{cases}$

Augmented matrix: $[A \mid b] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{array}\right]$

$\Updownarrow$ same info.

matrix presentation $= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$

$\Updownarrow$ same info.

vector presentation: $x_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ -7 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$

solved by elimination

two parameters

Solution set $= \{(-1-2s+3t,\ 4+s-2t,\ s,\ t) \mid s,t \in \mathbb{R}\}$

OR

Solution of (S): $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1-2s+3t \\ 4+s-2t \\ s \\ t \end{bmatrix}, \quad s,t \in \mathbb{R}$

OR

$$x = \begin{bmatrix} -1 \\ 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \quad s,t \in \mathbb{R}$$

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row operation via matrix multiplication

Defn: For the integer p, q, for $i = 1, 2, \cdots, p$
$j = 1, 2, \cdots, q$

define $E^{p,q}_{i,j}$ to be the $p \times q$ matrix

s.t. $\left(E^{p,q}_{i,j}\right)_{kl} = \begin{cases} 1 & \text{if } i=k,\ j=l \\ 0 & \text{otherwise} \end{cases}$

i.e.

eg: $E^{2,3}_{1,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $E^{3,3}_{2,1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, etc.

Lemma : Let A be $p \times q$ matrix,

then $E^{p,p}_{kl} A = p \times q$ matrix s.t.

the k-th row of $E^{pp}_{kl} A = [A_{l1}\ A_{l2} \cdots A_{lq}]$

and other row vanished.

Example : ① $E^{33}_{12} \overset{3\times4}{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

$$= \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

② $E^{33}_{11} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} A$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

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pf of lemma : $(E^{pp}_{kl} \overset{p\times q}{A})_{ij} = \sum_{g=1}^{p} (E^{pp}_{kl})_{ig} A_{gj}$

Case 1 : if $k \neq i$, then $(E^{pp}_{kl})_{ig} = 0 \quad \forall g = 1, 2, \ldots p$

$$\Rightarrow \left(E^{pp}_{kl} A\right)_{ij} = 0 \quad \text{if } i \neq k.$$

i.e. other than the k-th row, all vanishes.

<u>case 2</u> : if $k = i$

$$\left(E^{pp}_{kl} A\right)_{ij} = \sum_{q=1}^{p} \left(E^{pp}_{kl}\right)_{kq} A_{qj}$$

$$= \sum_{q=l} \left(E^{pp}_{kl}\right)_{kq} A_{qj} = A_{lj}$$

∴ on the k-th row, the j-th entry $= A_{lj}$ #

$E^{pp}_{kl} A$: <u>in term of row operation</u>

① multiply 0 to all row except the l-th row

② adding the l-th row to k-th row

③ multiply zero to l-th row.

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eg:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

① $0 \cdot R_1$, $0 \cdot R_3$

② $(1)R_2 + R_1$

③ $0 \cdot R_2$.

Lemma: Let A be $p \times q$ matrix, i, k are integer btw $1, p$.

(a). $\alpha R_i + R_k$: row operation of adding α multiple of i-th row to k-th row

is represented by $(I_p + \alpha E^{pp}_{ki})A$

(b) βR_i $\overset{\beta \neq 0}{\wedge}$ scalar multiple of i-th row

is represented by $(I_p + (\beta - 1) E^{pp}_{kk})A$

(c) $R_i \leftrightarrow R_k$: interchange of i-th and k-th row

is — by $(I_p - E^{pp}_{ii} + E^{pp}_{ik} - E^{pp}_{kk} + E^{pp}_{ki})A$.

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i.e. (a). $\begin{bmatrix} a_{11} & \cdots & \\ — & & a_{34} \end{bmatrix} = A \xrightarrow{2R_2 + R_1} \begin{bmatrix} 2a_{21} + a_{11} & 2a_{22} + a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ — & — & \end{bmatrix}$

$\|$

$$(I_3 + 2E^{33}_{12})A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

(b) $\begin{bmatrix} a_{11} & \cdots & \\ & & a_{34} \end{bmatrix} = A \xrightarrow{2R_2} \begin{bmatrix} a_{11} & \cdots & a_{14} \\ 2a_{21} & \cdots & 2a_{24} \\ a_{31} & & a_{34} \end{bmatrix}$

$\|$

$$I_3 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

(c). $$A \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Defn: The matrix M of $p \times p$ type is said to be a row operation of size p if

① $M = I_p + \alpha E_{ij}^{pp}$ $\quad (\alpha R_j + R_i)$

② $M = I_p + (\beta - 1) E_{kk}^{pp}$, $\beta \neq 0$ $\quad (\beta \cdot R_k)$

③ $M = I_p - E_{ii}^{pp} + E_{ik}^{pp} - E_{kk}^{pp} + E_{ki}^{pp}$. $\quad (R_k \leftrightarrow R_i)$

Thm: Matrix A is row equivalent to B iff

$\exists \{M_i\}_{i=1}^{N}$, seq of row operation matrix

s.t. $A = M_1 M_2 \cdots M_N B$.

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pf of lemma:

$\alpha R_i + R_k \Leftrightarrow (I_p + \alpha E_{ki}^{pp}) A$ ← p×q matrix

pf: (1) $\left((I_p + \alpha E_{ki}^{pp}) A\right)_{rs}$

if $r \neq k$, then

$$\sum_{m=1}^{p} \left(I_p + \alpha E_{ki}^{pp}\right)_{rm} A_{ms} = A_{rs}$$

i.e. r-th row remains unchanged.

if $r=k$, then

$$\sum_{m=1}^{p} \left(I_p + \alpha E_{ki}^{pp}\right)_{\cancel{r}\,k\,m} A_{ms} = A_{ks} + \alpha A_{is}$$

i.e. on the r-th row, the s-th entry

becomes $A_{rs}^{new} = \underbrace{A_{rs}}_{=\text{old}} + \alpha \underbrace{A_{is}}_{=\text{s-entry on i-th row.}}$

(2) $\beta R_k \longleftrightarrow \left(I_p + (\beta-1)E^{pp}_{kk}\right)A$

pf: $$\left(\left(I_p + (\beta-1)E^{pp}_{kk}\right)A\right)_{ij}$$

$$= A_{ij} + (\beta-1)\sum_{l=1}^{p} \left(E^{pp}_{kk}\right)_{il} A_{lj}$$

$$= \begin{cases} A_{ij} & \text{if } i \neq k. \\ \beta A_{ij} & \text{if } i = k, \end{cases}$$

i.e. the k-th row becomes β-multiple of original row.

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(3): $R_i \longleftrightarrow R_k$: $\left(I_p - E^{pp}_{ii} + E^{pp}_{ik} - E^{pp}_{kk} + E^{pp}_{ki}\right)A$

$$\left(\left(I_p - E^{pp}_{ii} + E^{pp}_{ik} - E^{pp}_{kk} + E^{pp}_{ki}\right)A\right)_{rs}$$

$$= A_{rs} - \delta_{ir}A_{is} + \delta_{ir}A_{ks} - \delta_{kr}A_{ks} + \delta_{kr}A_{is}$$

$\therefore$ if $r \neq i, k$, then R.H.S $= A_{rs} =$ original entry

i.e. row remains unchanged.

If $r = i \neq k$ (the new i-th row)

$$\text{RHS} = A_{is} - A_{is} + A_{ks} = A_{ks}$$

i.e. i-th row becomes the original k-th row.

If $r = k \neq i$,

$$\text{R.H.S} = A_{ks} - A_{ks} + A_{is} = A_{is}$$

i.e. the k-th row becomes the original i-th row.

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