Week 2 :

Recall from week 1 : learnt solving system of linear equation by Gaussian elimination

type of equation operations
- → switch
- → scalar multiplization
- → addition of rows.

Example :

$$S_0 : \begin{cases} x_1 - x_2 + x_3 = 2 \\ 3x_1 - 2x_2 + x_3 = 7 \\ -x_1 + 3x_2 - 5x_3 = 3 \end{cases}$$

$$\rightarrow S_1 : \begin{cases} x_1 - x_2 + x_3 = 2 \\ 0 + x_2 - 2x_2 = 1 \\ 0 + 2x_2 - 4x_3 = 5 \end{cases}$$

$$\rightarrow S_2 : \begin{cases} x_1 - x_2 + x_3 = 2 \\ 0 + x_2 - 2x_3 = 1 \\ 0 + 0 + 0 = 1 \end{cases}$$

No Addition

Only thing matter :

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 3 & -2 & 1 & 7 \\ -1 & 3 & -5 & 3 \end{array}\right)$$

Matrix.

polynomials

RHS

Defn: A $p \times q$ matrix is a $p \times q$ rectangular array

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1q} \\ c_{21} & \cdots & \boxed{c_{ij}} & \\ \vdots & & & \\ c_{p1} & \cdots & \cdots & c_{pq} \end{bmatrix} \Big\} \; p \text{ rows}$$

q columns

the (i,j)-th entry $= c_{ij}$

For $k = 1, 2, \cdots, p$ (OR n), the k-th row of $C = [c_{k1}\ c_{k2} \cdots c_{kq}]$

For $l = 1, 2, \cdots$ OR q, the l-th column of $C = \begin{bmatrix} c_{1l} \\ \vdots \\ c_{pl} \end{bmatrix}$

mean nothing mathematically, for convenience only.

Eg:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ \hline 3 & -2 & 1 & 7 \\ -1 & 3 & -5 & 3 \end{array}\right) \text{ (OR this one)} = C = \underset{p}{3} \times \underset{q}{4} \text{ matrix}$$

then $c_{11} = 1$, $c_{12} = -1$, $c_{13} = 1$, $c_{14} = 2$

$c_{21} = 3$, $c_{22} = -2$, $c_{23} = 1$, $c_{24} = 7$

$c_{31} = -1$, $c_{32} = 3$, $c_{33} = -5$, $c_{34} = 3$.

Example:

$$\left[\begin{array}{c|c|c} -1 & 2 & 5 \\ 4 & 0 & -6 \\ -4 & 2 & 2 \\ 2 & 5 & 6 \end{array}\right] = C = 4 \times 3 \text{ matrix.}$$

sometimes call them column vector.

then

$c_{11} = -1$ $\quad c_{12} = 2$ $\quad c_{13} = 5$

$c_{21} = 4$ $\quad c_{22} = 0$ $\quad c_{23} = -6$

$c_{31} = -4$ $\quad c_{32} = 2$ $\quad c_{33} = 2$

$c_{41} = 2$ $\quad c_{42} = 5$ $\quad c_{43} = 6$.

Defn: (coeff) Given a system of linear equations

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Coefficient Matrix $= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix} = m \times n$ matrix $= A$

vector of constant $= \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \vec{b}$, solution $= \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} = \vec{x}$.

write the system as matrix $[A|\vec{b}] = m \times (n+1)$ matrix

Called $[A|\vec{b}]$ as "Augmented matrix"

In the language of system of equation : Equation operation

In the language of matrix : row operations

Example:

$$\begin{cases} 2x_1 + 4x_2 - 3x_3 + 5x_4 + x_5 = 9 \\ 3x_1 + x_2 \qquad\quad x_4 - 3x_5 = 0 \\ -2x_1 + 7x_2 - 5x_3 + 2x_4 + 2x_5 = -3 \end{cases}$$

Row

coefficient matrix $A = \begin{bmatrix} 2 & 4 & -3 & 5 & 1 \\ 3 & 1 & 0 & 1 & -3 \\ -2 & 7 & -5 & 2 & 2 \end{bmatrix}$

Vector of constant $\vec{b} = \begin{bmatrix} 9 \\ 0 \\ -3 \end{bmatrix}$

The augmented matrix $= [A|b] = \left[\begin{array}{ccccc|c} 2 & 4 & -3 & 5 & 1 & 9 \\ 3 & 1 & 0 & 1 & -3 & 0 \\ -2 & 7 & -5 & 2 & 2 & -3 \end{array}\right]$

Defn: (Row-echelon form)

Let C be a $p \times q$ matrix

(i.e. $C = \overbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1q} \\ \vdots & & & \\ c_{p1} & \cdots & & c_{pq} \end{bmatrix}}^{q} \Big\} p$)

C is said to be (REF) if

① All row with only zeros are at the bottom

② Any non-zero row, the first non-zero element $= 1$.
that entry called leading ones of C.

③ The leading one is always strictly the right of that in row above

Eg: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, ...

Non-example: $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & \underset{\neq 1}{\boxed{4}} & 5 \end{bmatrix}$, $\begin{array}{c} \text{R.H.S} \\ \begin{bmatrix} 0 & \boxed{1} & 2 & 3 & 4 \\ \boxed{1} & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

Defn: A $p \times q$ matrix C is called reduced Row echelon form if

① C is REF

② the column consisting the leading one, is the only non-zero entry.

(we call such column, the pivot column of C)

More terminology: other column called free column

Example ①

free column

$\begin{bmatrix} \boxed{1} & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in RREF.

leading one

pivot column

Example ②

$$\begin{bmatrix} \boxed{1} & 0 & 5 & 3 & 0 & 0 & 5 \\ 0 & \boxed{1} & 3 & 6 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 3 \end{bmatrix}$$

free column

Leading one

pivot column

Rmk : Sometimes we use • $d_1, d_2, d_3, \cdots$ to denote the column index for pivot column

• $f_1, f_2, f_3, \cdots$ to denote the column index for free column

Example 1 : $d_1 = 1$, $d_2 = 3$, $d_3 = 6$

$f_1 = 2$, $f_2 = 4$, $f_3 = 5$

Example 2 : $d_1 = 1$, $d_2 = 2$, $d_3 = 5$, $d_4 = 6$

$f_1 = 3$, $f_2 = 4$, $f_3 = 7$.

Example 3 : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ No f_i

And $\begin{cases} d_1 = 1 \\ d_2 = 2 \\ d_3 = 3 \end{cases}$.

non-example: ①

$$\begin{bmatrix} \boxed{1} & 0 & 1 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

fail

B is REF But Not RREF

Defn: Let C, D be two $p \times q$ matrix,

If $\exists$ finite sequence of row operations s.t. D can be obtained by applying them to C, then we say that C and D are row equivalent

(rmk: Defn is analogy to that of system of linear equ.)

Thm Given a matrix A, $\exists!$ matrix B s.t.

① A and B are row equivalent

② B is in RREF.

pf: By Gaussian elimination, omitted (Later).

☆☆ Importance: canonical deformation of matrix picture: Given $A \xrightarrow{\text{transition}} B$ in RREF.

simplest possible!!

Example: find RREF of $A = \begin{bmatrix} 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 1 & 2 & 4 & 8 \\ 2 & 2 & 5 & 9 & 19 \end{bmatrix}$

Step 1: At 1st column, find the first non-zero entry: $a_{j1} \neq 0$, otherwise consider next column

in Eg: $j=3$: $a_{j1} = 1 \neq 0$. but $a_{11} = a_{21} = 0$.

Step 2: Swap the j-th row with 1-st row

$R_3 \Leftrightarrow R_1$: $\begin{bmatrix} 1 & 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 4 \\ 2 & 2 & 5 & 9 & 19 \end{bmatrix}$

Step 3: Apply scalar multiplication to 1st row s.t. a_{11} becomes 1.

(in this case, $a_{11} = 1$, $\rightsquigarrow$ trivially true)

Step 4: row operation to eliminate a_{j1}.

In this eg: $R_4 \longmapsto -2R_1 + R_4$

$$\begin{bmatrix} 1 & 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}$$

Step 5: (ignore 1st row and column)

$$\begin{bmatrix} 1 & 1 & 2 & 4 & 8 \\ 0 & 0 & \boxed{1} & 1 & 3 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}$$

focus!!

And repeat same procedure:

on the "1st" column, all are zeros!!

on the "2nd" column, 1st row

$R_3 \mapsto -R_2 + R_3$

$R_4 \mapsto -R_2 + R_4$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 2 & 4 & 8 \\ 0 & 0 & \boxed{1} & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

first pivot column

Hope: 2nd pivot column.

Step 6: Achieve "Hope"

$$\xrightarrow{R_1 \mapsto R_1 - 2R_2} \begin{bmatrix} 1 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & \boxed{0} & \boxed{1} \\ 0 & 0 & 0 & \boxed{0} & 0 \end{bmatrix}$$

first non-zero on 2nd column

focus

zero column

Step 7: Repeat Step 1 - 6 on $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Eliminate all non-zero element on the same column

$$\xrightarrow[R_2 \mapsto R_2 - 3R_3]{R_1 \mapsto R_1 - 3R_3} \begin{bmatrix} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \text{RREF}.$$

#

Example: find the RREF of $\begin{bmatrix} 0 & 0 & 2 & 2 & 6 & 2 & 3 \\ \boxed{2} & 4 & 1 & 3 & 7 & 3 & -1 \\ 1 & 2 & 2 & 3 & 4 & 2 & 1 \\ 1 & 2 & -1 & 0 & -1 & 2 & -1 \end{bmatrix}$

1st non-zero

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 4 & 1 & 3 & 7 & 3 & -1 \\ 0 & 0 & 2 & 2 & 6 & 2 & 3 \\ 1 & 2 & 2 & 3 & 4 & 2 & 1 \\ 1 & 2 & -1 & 0 & -1 & 2 & -1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{\text{(may)}} \begin{bmatrix} \boxed{1} & 2 & 2 & 3 & 4 & 2 & 1 \\ 0 & 0 & 2 & 2 & 6 & 2 & 3 \\ 2 & 4 & 1 & 3 & 7 & 3 & -1 \\ 1 & 2 & -1 & 0 & -1 & 2 & -1 \end{bmatrix}$$

Hope: becomes pivot column.

$$\xrightarrow[R_4 - R_1]{R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 2 & 3 & 8 & 2 & -1 \\ 0 & 0 & 2 & 2 & 6 & 2 & 3 \\ 0 & 0 & -3 & -3 & -9 & -1 & -3 \\ 0 & 0 & -3 & -3 & -9 & 0 & -2 \end{bmatrix}$$

Ignore

Goal: turn this to pivot column

$$\xrightarrow[R_3 + \frac{3}{2}R_2,\ R_4 + \frac{3}{2}R_2]{R_1 - R_2} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 0 & -4 \\ 0 & 0 & 2 & 2 & 6 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 2 & 3/2 \\ 0 & 0 & 0 & 0 & 0 & 3 & 5/2 \end{bmatrix}$$

$$\xrightarrow{R_2 \cdot \frac{1}{2}} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 1 & 3 & 1 & 3/2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 3/2 \\ 0 & 0 & 0 & 0 & 0 & 3 & 5/2 \end{bmatrix}$$

Ignore

Non-zero

$$\xrightarrow[R_4 - \frac{3}{2}R_3]{R_3 \cdot \frac{1}{2}} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 1 & 3 & 1 & 3/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = RREF.$$

pivot column.

Ex: find the solution to

$$\begin{cases} -7x - 6y - 12z = -33 \\ 5x + 5y + 7z = 24 \\ x + 4z = 5 \end{cases}$$

Step 1: Augmented matrix $= [A \mid b]$

$$= \left[\begin{array}{ccc|c} -7 & -6 & -12 & -33 \\ 5 & 5 & 7 & 24 \\ 1 & 0 & 4 & 5 \end{array}\right]$$

Step 2: find the RREF of $[A \mid b]$.

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 5 \\ 5 & 5 & 7 & 24 \\ -7 & -6 & -12 & -33 \end{array}\right]$$

$$\xrightarrow[R_3 + 7R_1]{R_2 - 5R_1} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 5 \\ 0 & 5 & -13 & -1 \\ 0 & -6 & 16 & 2 \end{array}\right]$$

$$\xrightarrow[\frac{1}{2}R_3]{\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 5 \\ 0 & 1 & -13/5 & -1/5 \\ 0 & -3 & 8 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 5 \\ 0 & 1 & -13/5 & -1/5 \\ 0 & 0 & 1/5 & 2/5 \end{array}\right]$$

$$\xrightarrow{5R_3} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 5 \\ 0 & 1 & -13/5 & -1/5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[R_1 - 4R_3]{R_2 + \frac{13}{5}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\therefore \begin{cases} x = -3 \\ y = 5 \\ z = 2 \end{cases}$$

Solution set $= \{(-3, 5, 2)\}$

$=$ unique sol.

Example: $\begin{cases} x - y + 2z = 1 \\ 2x + y + z = 8 \\ x + y = 5 \end{cases}$

Consider $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ 1 & 1 & 0 & 5 \end{array}\right]$

$$\xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 6 \\ 0 & 2 & -2 & 4 \end{array}\right]$$

$$\xrightarrow[R_2 \cdot \frac{1}{2}]{R_3 \cdot \frac{1}{3}} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

$$\xrightarrow[R_1 + R_2]{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ Solution set

$= \{(3 - t, 2 + t, t) \mid t \in \mathbb{R}\}$

Example: find the solution set to

$$\begin{cases} 2x + y + 7z - 7w = 2 \\ -3x + 4y - 5z - 6w = 3 \\ x + y + 4z - 5w = 2 \end{cases}$$

$$[A|b] = \left[\begin{array}{cccc|c} 2 & 1 & 7 & -7 & 2 \\ -3 & 4 & -5 & -6 & 3 \\ 1 & 1 & 4 & -5 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 1 & 4 & -5 & 2 \\ 2 & 1 & 7 & -7 & 2 \\ -3 & 4 & -5 & -6 & 3 \end{array}\right]$$

$$\xrightarrow[R_2 - 2R_1]{R_3 + R_1 \cdot 3} \left[\begin{array}{cccc|c} 1 & 1 & 4 & -5 & 2 \\ 0 & -1 & -1 & 3 & -2 \\ 0 & 7 & 7 & -21 & 9 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{cccc|c} 1 & 1 & 4 & -5 & 2 \\ 0 & 1 & 1 & -3 & 2 \\ 0 & 7 & 7 & -21 & 9 \end{array}\right]$$

$$\xrightarrow{R_3 - 7R_2} \left[\begin{array}{cccc|c} 1 & 1 & 4 & -5 & 2 \\ 0 & 1 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right] \neq \text{RREF.}$$

But we may stop here, impossible as a system.