Suggested Solution 8

(1) Let $f, g \in L^p(\mu)$, $1 < p < \infty$. Show that the function

$$\Phi(t) = \int_X |f + tg|^p \, d\mu$$

is differentiable at $t = 0$ and

$$\Phi'(0) = p \int_X |f|^{p-2} f g \, d\mu.$$

Hint: Use the convexity of $t \mapsto |f + tg|^p$ to get

$$|f + tg|^p - |f|^p \le t(|f + g|^p - |f|^p), \quad t > 0$$

and a similar estimate for $t < 0$.

Solution. Recall that for any convex function φ defined on $[0,1]$, one has the elementary inequality

$$\frac{\varphi(t) - \varphi(0)}{t - 0} \le \frac{\varphi(1) - \varphi(0)}{1 - 0}, \quad \forall t \in (0,1),$$

which could be deduced from the definition of convexity. For $p > 1, x \in X$, the function $\varphi(t) = |f(x) + tg(x)|^p$ is differentiable and convex whenever $f(x)$ and $g(x)$ are finite, which can be seen from $\varphi''(t) \ge 0$. Applying the inequality above to this particular convex function, We have

$$\frac{1}{t}\{|f + tg|^p - |f|^p\} \le |f + g|^p - |f|^p, \ \forall t \in (0,1).$$

By replacing t with $-t$, we obtain a similar inequality

$$|f|^p - |f - g|^p \le \frac{1}{t}\{|f + tg|^p - |f|^p\}, \ \forall t \in (-1,0).$$

Now the desired result follows from an application of Lebesgue's dominated convergence theorem.

(2) Suppose f is a measurable function on X, μ is a positive measure on X, and

$$\varphi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \quad (0 < p < \infty).$$

Let $E = \{p : \varphi(p) < \infty\}$. Assume $\|f\|_\infty > 0$.

(a) If $r < p < s$, $r \in E$, and $s \in E$, prove that $p \in E$.

(b) Prove that $\log \varphi$ is convex in the interior of E and that φ is continuous on E.

(c) By (a), E is connected. Is E necessarily open? Closed? Can E consist of a single point? Can E be any connected subset of $(0, \infty)$?

(d) If $r < p < s$, prove that $\|f\|_p \le \max(\|f\|_r, \|f\|_s)$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e) Assume that $\|f\|_r < \infty$ for some $r < \infty$ and prove that

$$\|f\|_p \to \|f\|_\infty \qquad \text{as } p \to \infty.$$

Solution.

(a) Write $p = \lambda r + (1-\lambda)s$ for $0 < \lambda < 1$. By Hölder's inequality,

$$\int_X |f|^p \, d\mu = \int_X |f|^{\lambda r} |f|(1-\lambda)s \, d\mu \le \left(\int_X |f|^r \, d\mu \right)^\lambda \left(\int_X |f|^s \, d\mu \right)^{1-\lambda},$$

which shows that φ is finite on $[r, z]$. It follows that E is an interval.

(b) Rewrite the inequality above as

$$\varphi(\lambda r + (1-\lambda)s) \le \varphi(r)^\lambda \cdot \varphi(s)^{1-\lambda}, \quad (0 < \lambda < 1).$$

It is also true for $\lambda = 0, 1$. Hence for all $\lambda \in [0, 1]$,

$$\log \varphi(\lambda r + (1-\lambda)s) \le \lambda \log \varphi(r) + (1-\lambda) \log \varphi(s).$$

since log is increasing. Thus $\log \varphi(p)$ is convex on $[r, s]$. Hence $\varphi(x)$ is continuous in the interior of E. It follows form monotonicity applying to $\chi_{|f|>1} f$ and $\chi_{|f|\le 1} f$ that $\varphi(x)$ is also continuous on ∂E.

(c) Let $X = (0, \infty)$ with the Lebesgue measure. E can be any connected subset of $(0, \infty)$. The basic functions to consider are of the form x^k and $x^k |\log x|^m$ near $x = 0$ and

$x = \infty$. Define

$$\begin{aligned} g_k(x) &= x^k \chi_{(0,1/2]}(x), \\ h_k(x) &= x^k \chi_{(2,\infty)}(x), \\ g_{k,m}(x) &= x^k |\log x|^m \chi_{(0,1/2]}(x), \\ h_{k,m}(x) &= x^k |\log x|^m \chi_{(2,\infty)}(x), \end{aligned}$$

It is easy to see that $\int_X g_k dx < \infty$ iff $k > -1$ and $\int_X h_k dx < \infty$ iff $k < -1$. Since $|\log x| \le C_\varepsilon e^{-\varepsilon}$ for $0 \le x \le 1$ and all $\epsilon > 0$, $\int_X g_{k,m} dx$ is finite for $k > -1$ and infinite for $k > -1$. For $k = -1$, direct computations by substituting $u = \log x$ yield

$$\int_X g_{k,m} dx = \int_0^{1/2} x^{-1} |\log x|^m dx = \int_{\log 2}^\infty u^m du,$$

which is finite iff $m < -1$. Similarly, one can show $\int_X h_{k,m} dx$ is finite for $k > -1$ and infinite for $k > -1$. If $k = -1$, the integral is finite if and only if $m < -1$. Note that $g_k^p = g_{pk}, g_{k,m}^p = g_{pk,pm}$ and similarly for h.

Now for $f = g_{-1,-2} + h_{-1,-2}$, one has $E = 1$. For $E = \emptyset$, take $f = g_{-1} + h_{-1}$. To get $E = (0, \infty)$, one may take $f = e^{-|x|}$. For $E = [1, p)$, take $f = g_{-1/p} + h_{-1,-2}$. Similarly it is easy to see that E can be any connected subset of $(0, \infty)$ for choosing f properly.

(d) From Q2(a), we have

$$\begin{aligned} \|f\|_p^p = \int_X |f|^p \le \left(\int_X |f|^r \right)^\lambda \left(\int_X |f|^s \right)^{1-\lambda} &= \|f\|_r^{r\lambda} \|f\|_s^{s(1-\lambda)} \\ &\le (\max \{\|f\|_r, \|f\|_s\})^{r\lambda} (\max \{\|f\|_r, \|f\|_s\})^{s(1-\lambda)} \\ &= \max \{\|f\|_r, \|f\|_s\}^p \end{aligned}$$

Obviously, if $\|f\|_r < \infty$ and $\|f\|_s < \infty$, then $\|f\|_p < \infty$. Thus $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e) Denote $E_a := \{x : a \le |f(x)|\}$ for every $0 < a < \|f\|_\infty$, then $0 < \mu(E_a) < \infty$. ($\|f\|_r < \infty$ implies $\mu(E_a) < \infty$.) Thus

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p} \ge \left(\int_{E_a} |f|^p \, d\mu \right)^{1/p} \ge a(\mu(E_a))^{1/p},$$

which implies $\varliminf_{p\to\infty} \|f\|_p \ge a$. Since a is arbitrary, we have $\varliminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty$.

On the other hand, for $p > r$,

$$\|f\|_p = \left(\int_X |f|^{p-r} |f|^r \, d\mu \right)^{1/p} \le \|f\|_r^{r/p} \|f\|_\infty^{1-r/p},$$

which implies $\overline{\lim_{p\to\infty}} \|f\|_p \le \|f\|_\infty$. In conclusion, we have

$$\lim_{p\to\infty} \|f\|_\infty = \|f\|_\infty.$$

(3) Assume, in addition to the hypothesis of the last problem, that

$$\mu(X) = 1.$$

(a) Prove that $\|f\|_r \le \|f\|_s$ if $0 < r < s \le \infty$.

(b) Under what conditions does it happen that $0 < r < s \le \infty$ and $\|f\|_r = \|f\|_s < \infty$?

(c) Prove that $L^r(\mu) \supset L^s(\mu)$ if $0 < r < s$. Under what conditions do these two spaces contain the same functions?

(d) Assume that $\|f\|_r < \infty$ for some $r > 0$, and prove that

$$\lim_{p\to 0} \|f\|_p = \exp\left\{ \int_X \log |f| \, d\mu \right\}$$

if $\exp\{-\infty\}$ is defined to be 0.

Solution.

(a) If $s < \infty$, the conclusion from from Hölder's inequality,

$$\int_X |f|^r \, d\mu \le \left(\int_X |f|^s \, d\mu \right)^{r/s} \left(\int_X 1 \, d\mu \right)^{1-r/s} = \|f\|_s^r.$$

If $s = \infty$, the desired result follows from

$$\|f\|_r \le \|f\|_\infty \left(\int_X 1 \, d\mu \right)^{1/r} = \|f\|_\infty.$$

(b) From the equality sign characterization in the Hölder inequality it is easy to see that $\|f\|_r = \|f\|_s < \infty$ if and only if $|f| = \|f\|_\infty < \infty$ a.e..

(c) We claim that under the condition $\mu(X) < \infty$, $L^r(\mu) = L^s(\mu)$ for $0 < r < s \le \infty$ if and only if the following property (call it L) holds:

There exists $\varepsilon_0 > 0$ such that for any measurable set $E \subset X$ with $\mu(E) > 0$ we have $\mu(E) > \varepsilon_0$.

In fact, if Property L holds, let $f \in L^r(\mu)$ and denote $E_n := \{x : |f| \geq n\}$. Then there exists $n_0 \in \mathbb{N}$ such that $\mu(E_{n_0}) = 0$ and thus $f \in L^\infty(\mu)$. Otherwise for all n, $\mu(E_n) > 0$. Thus $\mu(\{x : |f(x)| = \infty\}) \geq \lim_{n\to\infty} \mu(E_n) \geq \varepsilon_0$ and then $\|f\|_r = \infty$, a contradiction.

Conversely, suppose there is a sequence of measurable sets $\{E_n\}$ with $0 < \mu(E_n) < 3^{-n}$. Without loss of generality, E_n are mutually disjoint. Denote $a_n := \mu(E_n)$ and define

$$f = \begin{cases} \sum_{n=1}^{\infty} a_n^{-1/s} \chi_{E_n}, & \text{if } s < \infty, \\ \sum_{n=1}^{\infty} a_n^{-\frac{1}{2r}} \chi_{E_n}, & \text{if } s = \infty. \end{cases}$$

Then $f \in L^r$ but $f \notin L^s$. The proof is completed.

(d) Note $x - 1 - \log x \geq 0$ on $[0, \infty)$ implies that

$$\int_{\{|f|>1\}} \log |f| d\mu < \infty.$$

If $\mu(\{|f| = 0\}) > 0$, it suffices to proves the equality by showing $\lim_{p\to 0} \|f\|_p = 0$. There is a small $s > 1$, with s' be its conjugate s.t.

$$\begin{aligned} \|f\|_p &= \{\int_X |f|^p \chi_{\{|f|>0\}} d\mu\}^{\frac{1}{p}} \\ &\leq (\mu\{|f| > 0\})^{\frac{1}{s'p}} \|f\|_{sp} \text{ by Hölder inequality} \\ &\leq (\mu\{|f| > 0\})^{\frac{1}{s'p}} \|f\|_r \to 0 \text{ as } p \to 0 \end{aligned}$$

We may suppose $\infty > |f| > 0$ a.e. By Jensen's inequality, we have

$$\log \|f\|_p = \frac{1}{p} \log \int_X |f|^p \, d\mu \geq \frac{1}{p} \int_X \log |f|^p \, d\mu = \int_X \log |f| \, d\mu.$$

On the other hand, $x - 1 - \log x \geq 0$ on $[0, \infty)$ implies $\dfrac{\|f\|_p^p - 1}{p} \geq \log \|f\|_p$. Thus

$$\int_X \log |f| \, d\mu \leq \log \|f\|_p \leq \int_X \frac{|f|^p - 1}{p} \, d\mu$$

since $\mu(X) = 1$. Note that by convexity of the map $p \mapsto |f|^p$ we have $\dfrac{|f|^p - 1}{p}$ is increasing in p, which implies $\dfrac{|f|^p - 1}{p} \le \dfrac{|f|^r - 1}{r} \in L^1(\mu)$ and $\lim_{p\to 0} \dfrac{|f|^p - 1}{p} = \log |f|$. By Lebesgue's dominated convergence theorem for $|f| > 1$ and monotone convergence theorem for $|f| < 1$, we have

$$\lim_{p\to 0} \int_X \frac{|f|^p - 1}{p}\, d\mu = \lim_{p\to 0} \int_{\{|f|\ge 1\}} \frac{|f|^p - 1}{p}\, d\mu + \lim_{p\to 0} \int_{\{|f|<1\}} \frac{|f|^p - 1}{p}\, d\mu = \int_X \log|f|\, d\mu.$$

Thus by sandwich rule

$$\lim_{p\to 0} \|f\|_p = \exp\left\{ \int_X \log|f|\, d\mu \right\}$$

(4) For some measures, the relation $r < s$ implies $L^r(\mu) \subset L^s(\mu)$; for others, the inclusion is reversed; and there are some for which $L^r(\mu)$ does not contain $L^s(\mu)$ is $r \ne s$. Give examples of these situations, and find conditions on μ under which these situations will occur.

Solution.

First, we give examples of these situations:

(a) For $X = [0, 1]$ with usual Lebesgue measure, we have $L^r(\mu) \supset L^s(\mu)$ if $r < s$.

(b) For $X = \mathbb{N}$ with counting measure, we have $L^r(\mu) \subset L^s(\mu)$ if $r < s$.

(c) For $X = \mathbb{R}$ with usual Lebesgue measure, we have $L^r(\mu) \ne L^s(\mu)$ if $r \ne s$.

Second, we give simple conditions on μ under which these situations occur correspondingly:

(a) $\mu(X) < \infty$.

(b) Property L in 6(c) holds.

(c) $\mu(X) = \infty$ and Property L in 6(c) fails to hold.

(5) Suppose $\mu(\Omega) = 1$, and suppose f and g are positive measurable functions on Ω such that $fg \ge 1$. Prove that

$$\int_\Omega f\, d\mu \cdot \int_\Omega g\, d\mu \ge 1.$$

Solution. Since $fg \ge 1$, we have $\sqrt{fg} \ge 1$ and so by Hölder's inequality,

$$1 \le \int_\Omega \sqrt{f}\sqrt{g}\, d\mu \le \left(\int_\Omega f\, d\mu\right)^{1/2} \left(\int_\Omega g\, d\mu\right)^{1/2} = \left(\int_\Omega f\, d\mu \cdot \int_\Omega g\, d\mu\right)^{1/2}.$$

(6) Suppose $\mu(\Omega) = 1$ and $h : \Omega \to [0, \infty]$ is measurable. If

$$A = \int_\Omega h\, d\mu,$$

prove that

$$\sqrt{1+A^2} \leq \int_\Omega \sqrt{1+h^2}\, d\mu \leq 1+A.$$

If μ is Lebesgue measure on $[0,1]$ and if h is continuous, $h = f'$, the above inequalities have a simple geometric interpretation. From this, conjecture (for general Ω) under what conditions on h equality can hold in either of the above inequalities, and prove your conjecture.

Solution. The function $\phi(x) = \sqrt{1+x^2}$ is convex since its second derivative is always positive. Hence the first inequality follows from Jensen's inequality. The second equality follows from $|\Omega| = 1$ and $\sqrt{1+x^2} \leq 1+x$ for all $x \geq 0$.

In the case that $\Omega = [0,1]$ with μ the Lebesgue measure and $h = f'$ is continuous, then $\int_0^1 \sqrt{1+(f')^2}dx$ is the arc length of the graph of f. Then $A = f(1) - f(0)$. The first inequality means that the straight line is the shortest path while the second inequality means the longest path is the segment from $(0, f(0))$ to $(1, f(0))$ and then going up until $(1, f(1))$.

The intuition from this suggests that the second inequality is equality if and only if $h = 0, a.e.$, and the first inequality is equality if and only if $h = A, a.e.$ The first claim is clear since $\sqrt{1+x^2} = 1+x$ iff $x = 0$.If $h = A, a.e$, then trivially the first inequality holds. Conversely if the first inequality holds, it follows from an examination of the proof of Jensen's inequality that $\phi(A) = \phi(h(x)), a.e.$, so $h = A, a.e.$ since ϕ is injective on $[0,\infty)$.

(7) Optional. Suppose $1 < p < \infty$, $f \in L^p = L^p((0,\infty))$, relative to Lebesgue measure, and

$$F(x) = \frac{1}{x}\int_0^x f(t)\, dt \quad (0 < x < \infty).$$

(a) Prove Hardy's inequality

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p$$

which shows that the mapping $f \to F$ carries L^p into L^p.

(b) Prove that equality holds only if $f = 0$ a.e.

(c) Prove that the constant $\frac{p}{p-1}$ cannot be replaced by a smaller one.

(d) If $f > 0$ and $f \in L^1$, prove that $F \notin L^1$.

Suggestions: (a) Assume first that $f \geq 0$ and $f \in C_c((0,\infty))$. Integration by parts gives

$$\int_0^\infty F^p(x)\, dx = -p\int_0^\infty F^{p-1}(x) x F'(x)\, dx.$$

Note that $xF' = f - F$, and apply Hölder's inequality to $\displaystyle\int F^{p-1}f$. Then derive the general case.

(c) Take $f(x) = x^{-1/p}$ on $[1, A]$, $f(x) = 0$ elsewhere, for large A. See also Exercise 14, Chap. 8 in [R].

Solution. In fact we can show the inequality

$$\int_0^\infty |F|^p\,dx \le \frac{p}{p-1}\int_0^\infty |f||F|^{p-1}\,dx.$$

(a) $\vdash \|F\|_p \le \dfrac{p}{p-1}\|f\|_p, f \in \mathcal{L}^p(0,\infty), p \in (1,\infty)$

Let $f \in C_c(0,\infty), f \ge 0,$ first

$$\begin{aligned}\int_0^\infty F^p(x)dx &= xF^p(x)\big|_0^\infty - p\int_0^\infty F^{p-1}F'xdx \\ &= 0 - p\int_0^\infty F^{p-1}(f-F)dx,\end{aligned}$$

so

$$\int_0^\infty F^p(x)dx = \frac{p}{p-1}\int_0^\infty F^{p-1}fdx. \tag{1}$$

By Hölder's inequality,

$$\int_0^\infty F^p(x)dx \le \frac{p}{p-1}\{\int_0^\infty F^p(x)dx\}^{\frac{1}{q}}\|f\|_p$$

and (a) holds.

Now, for $f \in C_c(0,\infty)$, use

$$|F| \le \frac{1}{x}\int_0^x |f|$$

to get the same inequality.

Finally, for $f \in L^p(0,\infty)$, let $f_n \in C_c(0,\infty)$, $f_n \to f$ in L^p. Use an approximation argument to show $\{F_n\}$ is Cauchy and tends to F in $\mathcal{L}^p$ norm.

(b) $\vdash$ " $=$ " hold iff $f = 0$ a.e.

Let f satisfy

$$||F||_p = \frac{p}{p-1}\|f\|_p.$$

If f changes sign,

$$\widetilde{F}(x) = \frac{1}{x}\int_0^x |f| dt$$

$$\|\widetilde{F}\|_p > \|F\|_p = \frac{p}{p-1} = \||f|\|_p$$

Impossible. Therefore $f \geq 0$ say. By an approximation argument one can show that (1) holds for $f \geq 0,\ f \in L^p$. Following the proof in (a) one see by the equality condition in Hölder's inequality that $f^p =$ const $(F^{p-1})^q$, which implies there exists some positive constant c such that $F(x) = cf(x)$ a.e. Express this as an ODE for F and and solve it to get $f \equiv 0$ if $f \in L^p(0,\infty)$.

(c) Define

$$f(x) = \begin{cases} x^{-1/p}, & \text{if } x \in [1, A], \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|f\|_p = (\log A)^{1/p}$ and

$$F(x) = \begin{cases} 0, & \text{if } x \in (0,1), \\ \dfrac{p}{p-1}\left(x^{-\frac{1}{p}} - x^{-1}\right), & \text{if } x \in [1, A], \\ \dfrac{p}{p-1}\left(A^{1-\frac{1}{p}} - 1\right)x^{-1}, & \text{if } x \in (A, \infty). \end{cases}$$

Then $\|F\|_p^p = I_1 + I_2$, where

$$\begin{aligned} I_1 &= \int_1^A \left(\frac{p}{p-1}\left(x^{-\frac{1}{p}} - x^{-1}\right)\right)^p dx \\ &= \left(\frac{p}{p-1}\right)^p \int_1^A \left(x^{-\frac{1}{p}} - x^{-1}\right)^p dx \\ I_2 &= \int_A^\infty \left(\frac{p}{p-1}\left(A^{1-\frac{1}{p}} - 1\right)x^{-1}\right)^p dx \\ &= \frac{p^p}{(p-1)^{p+1}}\left(1 - A^{\frac{1}{p}-1}\right)^p dx. \end{aligned}$$

Suppose on the contrary that the constant $\dfrac{p}{p-1}$ can be replaced by $\dfrac{\gamma p}{p-1}$ for some $\gamma \in (0,1)$. Then there exists $\delta \in (\gamma, 1)$. Note that there exists $A_0 > 1$ such that for

$x > A_0$, $x^{-\frac{1}{p}} - x^{-1} > \delta x^{-\frac{1}{p}}$. Then for sufficiently large $A > A_0$,

$$\begin{aligned} I_1 &> \frac{\delta p}{p-1} \int_{A_0}^{A} x^{-1}\,dx \\ &= \frac{\delta p}{p-1} (\log A - \log A_0) \\ &> \frac{\gamma p}{p-1} \log A \\ &= \frac{\gamma p}{p-1} \|f\|_p^p . \end{aligned}$$

This implies $\|F\|_p > \dfrac{\gamma p}{p-1} \|p\|_f$ if A is sufficiently large. Contradiction arises.

(d) Since $f > 0$ on $(0, \infty)$, there exists $x_0 > 0$ such that $c_0 := \displaystyle\int_0^{x_0} f(t)\,dt$. Then

$$\int_{x_0}^{\infty} F(x)\,dx = \int_{x_0}^{\infty} \frac{1}{x} \int_0^{x} f(t)\,dtdx \geq \int_{x_0}^{\infty} \frac{1}{x} \int_0^{x_0} f\,dtdx \geq \int_{x_0}^{\infty} \frac{c_0}{x}\,dx = \infty,$$

showing that $F \notin L^1$.