Suggested Solution 4

1. We continue our study of the Lebesgue measure beginning in Ex 3. Show that

   (a) $\mathcal{L}^n$ is a Borel measure.

   (b) For every set E, there exists a sequence of open sets $\{G_k\}$ satisfying $E \subset G_k$ and
$$\mathcal{L}^n(E) = \lim_{k\to\infty} \mathcal{L}^n(G_k)\ .$$

   (c) For every measurable set A, there exists a sequence of compact sets $\{K_j\}$ satisfying $K_j \subset A$ and
$$\mathcal{L}^n(A) = \lim_{j\to\infty} \mathcal{L}^n(K_j)\ .$$
   Hint: First assume A is bounded.

Solution. Recall that the Lebesgue measure is given by
$$\mathcal{L}^n(E) = \inf\left\{\sum_k |C_k| : \ E \subset \bigcup_k C_k, \ C_k \text{ closed cubes}\right\}\ .$$
Since every cube can be subdivided into finitely many cubes with arbitrary small diameter, we also have
$$\mathcal{L}^n(E) = \inf\left\{\sum_k |C_k| : \ E \subset \bigcup_k C_k, \ C_k \text{ closed cubes}\ , \text{diam } C_k \le \delta\right\}\ ,$$
for each $\delta > 0$.

(a). Let $2\rho = \text{dist }(A, B) > 0$ for two subsets A and B. If we could establish
$$\mathcal{L}^n(A \cup B) = \mathcal{L}^n(A) + \mathcal{L}^n(B),$$
Caratheodory's criterion will show that $\mathcal{L}^n$ is Borel. Indeed, let $\{C_k\}$ be a collection of cube of diameter less than ρ covering $A \cup B$ so that $\mathcal{L}^n(A\cup B) + \varepsilon \ge \sum_k |C_k|$. By our choice of ρ, no C_k can intersect A and B simultaneously. We can divide all cubes into three classes: $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ respectively for those intersect A only, intersect B only and intersect none of A or B. Then $A \subset \cup_{\mathcal{C}_1} C_k$ and $B \subset \cup_{\mathcal{C}_2} C_k$ so
$$\mathcal{L}^n(A\cup B) + \varepsilon \ge \sum_k |C_k| \ge \sum_{\mathcal{C}_1} |C_k| + \sum_{\mathcal{C}_2} |C_k| \ge \mathcal{L}^n(A) + \mathcal{L}^n(B)\ ,$$
done after letting $\varepsilon \to 0$.

(b). Alternatively we have
$$\mathcal{L}^n(E) = \inf\left\{\sum_k |C_k| : \ E \subset \bigcup_k C_k, \ C_k \text{ open cubes}\right\}\ .$$
The result follows easily.

(c). We decompose $\mathbb{R}^n = \bigcup_k E_k$ where $E_1 = \overline{B_1(0)}, E_k = \{x : \ k-1 \le |x| \le k\}$ for $k \ge 2$. For a measurable A, let $A_k = A \cap E_k$. Since A'_k is measurable, we can find an open G_k containing A'_k such that $\mathcal{L}^n(G_k \setminus A'_k) < \varepsilon/2^k$. So $\mathcal{L}^n(A_k \setminus K'_k) < \varepsilon/2^k$ where $K_k = E_k \cap G_k$. When $\mathcal{L}^n(A) < \infty$, we can fix some large N such that $\sum_N \mathcal{L}^n(A_k) < \varepsilon$. Then $K = \cap_{k=1}^{N-1} K_k$ satisfies $K \subset A$ and $\mathcal{L}^n(A \setminus K) < 2\varepsilon$. If $\mathcal{L}^n(A) = \infty$, for each $M > 0$, we can find some large N such that $\sum_{k=1}^N \mathcal{L}^n(A_k) > M + 1$. Then $K = \sum_{k=1}^{N-1} K_k$ satisfies $\mathcal{L}^n(K) > M$.

2. Let $(\mathbb{R}^n, \mathcal{B}, \mu)$ be a measure space where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^n$. Suppose that μ is translational invariant, i.e., $\mu(E+x) = \mu(E)$, $\forall x \in \mathbb{R}^n$, $E \in \mathcal{B}$, and that μ is non-trivial in the sense that $0 < \mu([0,1]^n) < \infty$. Show that μ is a constant multiple of the Lebesgue measure on $\mathcal{R}^n$ when restricted to $\mathcal{B}$.

   Solution. Let $\lambda = \mu([0,1]^n)^{-1}\mu$ so $\lambda(C) = 1$ for all unit cubes C. We need to show $\lambda = \mathcal{L}^n$. Using translational invariance and the decomposition of an open set into a countable union of almost disjoint cubes of various diameter, one can show that λ and $\mathcal{L}^n$ are equal on open sets. Let E be a bounded Borel set. It is easy to see that both $\mathcal{L}^n(E)$ and $\lambda(E)$ are finite. By the regularity of the Lebesgue measure, for each $\varepsilon > 0$, there are open and closed sets, G, F, $F \subset E \subset G$, such that $\lambda(G \setminus F) = \mathcal{L}^n(G \setminus F) < \varepsilon$. Then $\lambda(G \setminus E) < \varepsilon$ which implies

$$\mathcal{L}^n(E) \le \mathcal{L}^n(G) \le \lambda(G) \le \lambda(F) + \varepsilon \le \lambda(E) + \varepsilon.$$

   Similarly, $\lambda(E \setminus F) < \varepsilon$ implies $\lambda(E) \le \mathcal{L}^n(E) + \varepsilon$. Putting things together we get it.

3. Let X be a metric space and $\mathcal{C}$ be a subset of $\mathcal{P}_X$ containing the empty set and X. Assume that there is a function $\rho : \mathcal{C} \to [0, \infty]$ satisfying $\rho(\phi) = 0$. For each $\delta > 0$, show that (a)

$$\mu_\delta(E) = \inf\Big\{ \sum_k \rho(C_k) : E \subset \bigcup_k C_k, \quad \text{diam } (C_k) \le \delta \Big\}$$

   is an outer measure on X, and (b) $\mu(E) = \lim_{\delta \to 0} \mu_\delta(E)$ exists and is also an outer measure on X.

   Solution:

   (a) To see that (i) is satisfied, observe that the empty set ϕ is contained in any $\mathcal{C}$, so that $\mu_\delta(\phi) = \rho(\phi) = 0$. Let $A \subset \bigcup_{j=1}^\infty A_j$. Suppose that $\sum_j \mu_\delta(A_j) < \infty$ (otherwise there is nothing to prove). For each $\varepsilon > 0$, we can find $C_k^j, k \ge 1$, in $\mathcal{C}$ such that diameter$(C_k^j) \le \delta, A_j \subset \bigcup_k C_k^j$ and $\sum_k \varphi(C_k^j) \le \mu_\delta(A_j) + \varepsilon/2^j$. As $\{C_k^j\}$ covers A and diameter$(C_k^j) \le \delta$,

$$\begin{aligned} \mu_\delta(A) &\le \sum_{j,k} \varphi(C_k^j) \\ &\le \sum_j \sum_k \varphi(C_k^j) \\ &\le \sum_j \left(\mu_\delta(A_j) + \frac{\varepsilon}{2^j} \right) \\ &\le \sum_j \mu_\delta(A_j) + \varepsilon, \end{aligned}$$

   and (ii) holds after letting ε tend to 0.

   (b) Since $\mu_\delta(E)$ is increasing in δ, $\mu(E) = \lim_{\delta \to 0} \mu_\delta(E)$ exists. Since $\mu_\delta(\phi) = 0, \mu(\phi) = 0$. Hence (i) is satisfied. Let $A \subset \bigcup_{j=1}^\infty A_j$. Suppose that $\sum_j \mu(A_j) < \infty$. One has

$$\mu(A) \le \mu_\delta(A) \le \sum_j \mu_\delta(A_j).$$

   It follows from monotonicity that

$$\mu(A) \le \liminf_{\delta \to 0} \sum_j \mu_\delta(A_j) = \sum_j \mu(A_j).$$

4. Consider in the previous problem the Euclidean space $\mathbb{R}^n, \mathcal{C} = \mathcal{P}_X$ and $s \in [0, \infty)$. Let

$$\rho(C) = (\text{diam } (C))^s \ ,$$

where the diameter of C is given by $\sup_{x,y \in C} |x - y|$. Show that the outer measure $\mu(E) = \lim_{\delta \to 0} \mu_\delta(E)$ is Borel.

Solution. Similar to the proof in 1(a) using the fact that $\delta \to 0$. The resulting measure is the unnormalized s-Hausdorff measure.

5. Let X be a metric space and $C(X)$ the collection of all continuous real-valued functions in X. Let $\mathcal{A}$ consist of all sets of the form $f^{-1}(G)$ which $f \in C(X)$ and G is open in $\mathbb{R}$. The "Baire σ-algebra" is the σ-algebra generated by $\mathcal{A}$. Show that the Baire σ-algebra coincides with the Borel σ-algebra $\mathcal{B}$.

Solution: It is clear that the Baire σ-algebra is contained in $\mathcal{B}$, since $f^{-1}(G)$ is an open whenever f is continuous and G is open in $\mathbb{R}$. Conversely, We prove the inclusion using the metric d function on the metric space X. Let U be arbitrary open set in X, F be its complement in X, and consider the function $f(x) \equiv \inf\{d(x, y) : y \in F\}$. As F is closed, $f(x)$=0 $\Leftrightarrow$ x is not in U, obviously f is in $C(X)$ and $U = f^{-1}(\mathbb{R}^+)$. Hence every open U is in the Baire σ-algebra. This exercise shows that in order to make all continuous functions measurable, the smallest sigma algebra is the Borel one.

6. Identify the Riesz measures corresponding to the following positive functionals ($X = \mathbb{R}$):

   (a) $\Lambda_1 f = \displaystyle\int_a^b f \, dx$, and

   (b) $\Lambda_2 f = f(0)$.

Solution:

   (a) $\mu_1 =$ the restriction of the Lebesgue measure on $[a, b]$.
   $\mu_1(E) = \mathcal{L}^1(E \cap [a, b])$

   (b) The Dirac delta measure at 0.

7. Let c be the counting measure on $\mathbb{R}$,

$$c(A) = \begin{cases} \#A, & A \neq \phi, \\ 0, & A = \phi. \end{cases}$$

Is there a positive functional

$$\Lambda f = \int f \, dc \quad ?$$

Solution: No, let f(x) be a non-negative continuous function of compact support that is 1 for all x in $[0, 1]$ and decreases to zero outside the interval,

$$\int f dc \geq \int \chi_{[0,1]} dc = \infty.$$