Suggested Solution 10

(1) Let $\mathcal{L}^1$ be the Lebesgue measure on $(0,1)$ and μ the counting measure on $(0,1)$. Show that $\mathcal{L}^1 \ll \mu$ but there is no $h \in L^1(\mu)$ such that $d\mathcal{L}^1 = h\,d\mu$. Why?

Solution. If $\mu(E) = 0$, then $E = \phi$, which implies $\mathcal{L}^1(E) = 0$. Hence, $\mathcal{L}^1 \ll \mu$.

Suppose on the contrary, that $\exists h \in L^1(\mu)$ such that $d\mathcal{L}^1 = \int h\,d\mu$. Since $h \in L^1(\mu)$, $h = 0$ except on a countable set. It follows that $\mathcal{L}^1(\{h = 0\}) = 1$. However,

$$\mathcal{L}^1(\{h=0\}) = \int_{\{h=0\}} h\,d\mu = 0.$$

This is a contradiction. Radon-Nikodym theorem does not apply here because μ is not σ-finite.

(2) Let μ be a measure and λ a signed measure on $(X, \mathfrak{M})$. Show that $\lambda \ll \mu$ if and only if $\forall \varepsilon > 0$, there is some $\delta > 0$ such that $|\lambda(E)| < \varepsilon$ whenever $|\mu(E)| < \delta$, $\forall E \in \mathfrak{M}$.

Solution. ($\Leftarrow$) Suppose $\mu(E) = 0$. By the hypothesis, for all $\varepsilon > 0$, $|\lambda(E)| < \varepsilon$. This implies $\lambda(E) = 0$, hence $\lambda \ll \mu$.

($\Rightarrow$) Suppose on the contrary that $\exists \varepsilon_0 > 0$ such that $\forall n \in \mathbb{N}$, $\exists E_n \in \mathfrak{M}$ with $\mu(E_n) < 2^{-n}$ such that $\lambda(E_n) \geq \varepsilon_0$. Put $E = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} E_k$. Then $\mu(E) = 0$ but

$$\lambda(E) = \lim_{n\to\infty} \lambda\left(\bigcup_{k\geq n} E_k\right) \geq \varepsilon_0 > 0.$$

This contradicts the fact that $\lambda \ll \mu$.

(3) Let μ be a σ-finite measure and λ a signed measure on $(X, \mathfrak{M})$ satisfying $\lambda \ll \mu$. Show that

$$\int f\,d\lambda = \int fh\,d\mu, \quad \forall f \in L^1(\lambda),\ fh \in L^1(\mu)$$

where $h = \dfrac{d\lambda}{d\mu} \in L^1(\mu)$.

Solution.

Step 1. $f = \chi_E$ for some $E \in \mathfrak{M}$.

We have

$$\int_X \chi_E\,d\lambda = \lambda(E) = \int_E h\,d\mu = \int_X \chi_E h\,d\mu.$$

Step 2. f is a simple function.

This follows directly from Step 1.

Step 3. $f \geq 0$ is measurable.

Pick $0 \leq s_n \nearrow f$. Then $0 \leq s_n h \nearrow fh$ on $\{h \geq 0\}$ and $0 \leq -s_n h \nearrow -fh$ on $\{h < 0\}$. Hence,

$$\begin{aligned}
\int_X f\,d\lambda &= \int_{h\geq 0} f\,d\lambda - \int_{h<0} -f\,d\lambda \\
&= \sup_{0\leq s\leq f} \int_{h\geq 0} s\,d\lambda - \sup_{0\leq s\leq f} \int_{h<0} -s\,d\lambda \\
&= \sup_{0\leq s\leq f} \int_{h\geq 0} sh\,d\mu - \sup_{0\leq s\leq f} \int_{h<0} -sh\,d\mu \text{ (by Step 2)} \\
&= \int_{h\geq 0} fh_+\,d\mu - \int_{h<0} fh_-\,d\mu \\
&= \int_X f(h_+ - h_-)\,d\mu \\
&= \int_X fh\,d\mu.
\end{aligned}$$

Step 4. $f \in L^1(\lambda)$.

Writing $f = f_+ - f_-$, the result follows from Step 3.

(4) Let μ, λ and ν be finite measures, $\mu \gg \lambda \gg \nu$. Show that $\dfrac{d\nu}{d\mu} = \dfrac{d\nu}{d\lambda}\dfrac{d\lambda}{d\mu}$, μ a.e.

Solution. By Radon-Nikodym, for all measurable sets E,

$$\nu(E) = \int_E f d\lambda, \quad \lambda(E) = \int_E g d\mu, \quad \nu(E) = \int_E h d\mu \ ,$$

for some non-negative measurable f, g and h. By simple function approximation, it follows that

$$\int \varphi d\nu = \int \varphi f d\lambda, \quad \int \varphi d\lambda = \int \varphi g d\mu, \quad \int \varphi d\nu = \int \varphi h d\mu \ ,$$

hold for all measurable φ. We have

$$\int \varphi d\nu = \int h\varphi d\mu \ ,$$

and, on the other hand,

$$\int \varphi d\nu = \int \varphi f d\lambda = \int \varphi f g d\mu \ .$$

By comparison, $h = fg$ a.e.

(5) Show that the completion of $C_c(X)$ under the sup-norm is $C_0(X)$ where X is a locally compact, Hausdorff space.

Solution. We regard $C_c(X)$ as a subspace in the Banach space $C_b(S)$ consisting of all bounded, continuous functions on X. We take it a known fact that it is a Banach space under the supnorm. First we show C_0 contains the closure of C_c. Let $f \in \overline{C_c(X)}$, for $\varepsilon > 0$, there is $g \in C_c(X)$ such that $\|f-g\|_\infty < \varepsilon$. If we take K to be the support of g. Then $|f|$ is less than ε outside K. On the other hand, if $f \in C_0(X)$, there is some compact F such that $|f| < \varepsilon$ outside F. Let φ be a continuous function with compact support, $0 \le \varphi \le 1$, $\varphi = 1$ on F. The existence of φ is ensured by the topological assumption on X. Then the function $h = f\varphi \in C_c(X)$ and satisfies $\|f-h\|_\infty < \varepsilon$. We have shown that $C_0(X)$ is the closure of $C_c(X)$ in the space $C_b(X)$.

(6) Provide a proof of Proposition 5.8.

Solution.

(a) Let $E = \overset{\circ}{\bigcup} E_j \in \mathfrak{M}$. If λ is concentrated on A, then $\lambda(E_j) = \lambda(E_j \cap A)$, and so

$$\begin{aligned}
|\lambda|(E) &= \sup\{\sum |\lambda(E_j)| : \ E = \overset{\circ}{\bigcup} E_j, \ E_j \in \mathfrak{M}\} \\
&= \sup\{\sum |\lambda(E_j \cap A)| : \ E \cap A = \overset{\circ}{\bigcup} (E_j \cap A), \ E_j \in \mathfrak{M}\} \\
&= |\lambda|(E \cap A).
\end{aligned}$$

(b) If $\lambda_1 \perp \lambda_2$, then λ_j is concentrated on some A_j $(j = 1, 2)$ with $A_1 \cap A_2 = \emptyset$. By part (a), $|\lambda_j|$ is concentrated on A_j. Therefore, $|\lambda_1| \perp \||\lambda_2|$.

(c) Suppose μ is concentrated on A. If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1(A) = \lambda_2(A) = 0$, which implies $(\lambda_1 + \lambda_2)(A) = 0$. Hence, $\lambda_1 + \lambda_2 \perp \mu$.

(d) Suppose $\mu(E) = 0$. If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1(E) = \lambda_2(E) = 0$, which implies $(\lambda_1 + \lambda_2)(E) = 0$. Hence, $\lambda_1 + \lambda_2 \ll \mu$.

(e) Let $E = \overset{\circ}{\bigcup} E_j$ and suppose $\mu(E) = 0$. Then $E_j \subset E$ implies $\mu(E_j) = 0$. If $\lambda \ll \mu$, then $\lambda(E_j) = 0$. Therefore, $\sum |\lambda(E_j)| = 0$ and it follows that $|\lambda|(E) = 0$.

(f) Suppose λ_2 is concentrated on A. If $\lambda_2 \perp \mu$, then $\mu(A) = 0$, which implies $\lambda_1(A) = 0$ by $\lambda_1 \ll \mu$. Hence, $\lambda_1 \perp \lambda_2$.

(g) By part (f), $\lambda \perp \lambda$. This is impossible unless $\lambda = 0$.

(7) Show that $M(X)$, the space of all signed measures defined on $(X, \mathfrak{M})$, forms a Banach space under the norm $\|\mu\| = |\mu|(X)$.

Solution. It is clear that the $M(X)$ is a normed vector space if the norm is defined as in the question.

Recall the fact that a normed vector space is a Banach space if and only if every absolutely summable sequence is summable. Let $\{\mu_k\}$ be an absolutely summable sequence. Let E be a measurable set. We immediately have

$$\sum_{k=1}^{\infty} |\mu_k(E)| \leq \sum_{k=1}^{\infty} |\mu_k|(E) \leq \sum_{k=1}^{\infty} |\mu_k|(X) < \infty,$$

hence $\sum \mu_k(E)$ converges absolutely. $\forall E \in \mathfrak{M}$, put

$$\mu(E) = \sum_{k=1}^{\infty} \mu_k(E)$$

which exists as a real number by the above argument. We will prove the countable additivity. Let E_n be a sequence of pairwise disjoint measurable sets. Then

$$\begin{aligned}
\mu\left(\bigcup E_n\right) &= \sum_{k=1}^{\infty} \mu_k\left(\bigcup E_n\right) \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \mu_k(E_n) \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu_k(E_n) \text{ (by absolute convergence)} \\
&= \sum_{n=1}^{\infty} \mu(E_n).
\end{aligned}$$

We have proved that μ is a signed measure. To show that μ_n converges to μ in $\|\cdot\|$, let X_n be a partition of X.

$$\begin{aligned}
\sum_{n=1}^{\infty} \left| \left(\mu - \sum_{k=1}^{m} \mu_k\right)(X_n) \right| &= \left| \sum_{n=1}^{\infty}\sum_{k=m}^{\infty} \mu_k(X_n) \right| \\
&\leq \sum_{k=m}^{\infty}\sum_{n=1}^{\infty} |\mu_k(X_n)| \\
&\leq \sum_{k=m}^{\infty} |\mu_k|(X) = \sum_{k=m}^{\infty} \|\mu_k\| \to 0
\end{aligned}$$

so that $\left\| \sum \mu_k - \mu \right\| \to 0$ as $k \to \infty$.

(8) Show that $M_r(X)$ is a closed subspace in $M(X)$ on $(X, \mathcal{B})$ where X is a locally compact Hausdorff space. Hence it is a Banach space.

Solution. It is routine.