Exercise 8

Standard notations are in force. Many problems are taken from [R].

(1)

$$\Phi(t) = \int_X |f + tg|^p \, d\mu$$

is differentiable at t = 0 and

$$\Phi'(0) = p \int_X |f|^{p-2} fg \, d\mu.$$

Hint: Use the convexity of $t\mapsto |f+tg|^p$ to get

$$|f + tg|^p - |f|^p \le t(|f + g|^p - |f|^p), \quad t > 0$$

and a similar estimate for t < 0.

(2) Suppose f is a measurable function on X, μ is a positive measure on X, and

$$\varphi(p) = \int_{X} |f|^{p} d\mu = ||f||_{p}^{p} \quad (0$$

Let $E = \{p : \varphi(p) < \infty\}$. Assume $\|f\|_{\infty} > 0$.

- (a) If $r , <math>r \in E$, and $s \in E$, prove that $p \in E$.
- (b) Prove that $\log \varphi$ is convex in the interior of E and that φ is continuous on E.
- (c) By (a), E is connected. Is E necessarily open? Closed? Can E consist of a single point? Can E be any connected subset of $(0, \infty)$?
- (d) If $r , prove that <math>||f||_p \le \max(||f||_r, ||f||_s)$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.
- (e) Assume that $||f||_r < \infty$ for some $r < \infty$ and prove that

$$||f||_p \to ||f||_\infty$$
 as $p \to \infty$.

(3) Assume, in addition to the hypothesis of the previous problem, that

$$\mu(X) = 1.$$

(a) Prove that $||f||_r \le ||f||_s$ if $0 < r < s \le \infty$.

- (b) Under what conditions does it happen that $0 < r < s \le \infty$ and $||f||_r = ||f||_s < \infty$?
- (c) Prove that $L^r(\mu) \supset L^s(\mu)$ if 0 < r < s. Under what conditions do these two spaces contain the same functions?
- (d) Assume that $||f||_r < \infty$ for some r > 0, and prove that

$$\lim_{p \to 0} \|f\|_p = \exp\left\{ \int_X \log|f| \, d\mu \right\}$$

if $\exp\{-\infty\}$ is defined to be 0.

- (4) For some measures, the relation r < s implies $L^r(\mu) \subset L^s(\mu)$; for others, the inclusion is reversed; and there are some for which $L^r(\mu)$ does not contain $L^s(\mu)$ is $r \neq s$. Give examples of these situations, and find conditions on μ under which these situations will occur.
- (5) Suppose $\mu(\Omega) = 1$, and suppose f and g are positive measurable functions on Ω such that $fg \geq 1$. Prove that

$$\int_{\Omega} f \, d\mu \cdot \int_{\Omega} g \, d\mu \ge 1.$$

(6) Suppose $\mu(\Omega) = 1$ and $h: \Omega \to [0, \infty]$ is measurable. If

$$A = \int_{\Omega} h \, d\mu,$$

prove that

$$\sqrt{1+A^2} \le \int_{\Omega} \sqrt{1+h^2} \, d\mu \le 1+A.$$

If μ is Lebesgue measure on [0, 1] and if h is continuous, h = f', the above inequalities have a simple geometric interpretation. From this, conjecture (for general Ω) under what conditions on h equality can hold in either of the above inequalities, and prove your conjecture.

(7) Optional. Suppose $1 , <math>f \in L^p = L^p((0, \infty))$, relative to Lebesgue measure, and

$$F(x) = \frac{1}{x} \int_0^x f(t) dt \quad (0 < x < \infty).$$

(a) Prove Hardy's inequality

$$||F||_p \le \frac{p}{p-1} ||f||_p$$

which shows that the mapping $f \to F$ carries L^p into L^p .

(b) Prove that equality holds only if f = 0 a.e.

- (c) Prove that the constant $\frac{p}{p-1}$ cannot be replaced by a smaller one.
- (d) If f > 0 and $f \in L^1$, prove that $F \notin L^1$.

Suggestions: (a) Assume first that $f \geq 0$ and $f \in C_c((0,\infty))$. Integration by parts gives

$$\int_0^\infty F^p(x) dx = -p \int_0^\infty F^{p-1}(x) x F'(x) dx.$$

Note that xF' = f - F, and apply Hölder's inequality to $\int F^{p-1}f$. Then derive the general case.

(c) Take $f(x) = x^{-1/p}$ on [1, A], f(x) = 0 elsewhere, for large A. See also Exercise 14, Chap. 8 in [R].