Chapter 5

Radon-Nikodym Theorem

Signed measures come up in at least two occasions. First, for any non-negative μ-measurable function f, the map $E \mapsto \int_E f d\mu$ is a measure. When f changes sign, this map is still countably additive and it should be a "signed measure". Second, in Theorem 2.8 it is shown that every positive linear functional on $C_c(X)$ (X is a locally compact Hausdorff space) comes from a regular Borel measure. From the viewpoint of functional analysis, it is desirable to characterize the dual space of $C_c(X)$ as the space of "signed measures". We introduce signed measures in Section 1. Just like the absolute value of a function is non-negative, it is shown that the " absolute value " of a signed measure, its total variation, is a measure. In Section 2 we establish the important theorem of Radon-Nikodym and discuss how to decompose a measure or a signed measure into its absolute continuous and singular parts with respect to another measure. Then we use this theorem to establish the full Riesz representation theorem in Section 3. Weak* convergence of sequences of signed measures is discussed in Section 4. As an application of the Riesz representation theorem we give a characterization of weakly convergent L^1-sequences, part of the Dunford-Pettis theorem. Finally, as another application of the Riesz representation theorem, we prove Herglotz-Riesz theorem concerning the boundary trace of a non-negative harmonic function in Section 5.

5.1 Signed Measures

Consider $\mathcal{M}$ a σ-algebra on the non-empty set X. A map $\mu : \mathcal{M} \to \mathbb{R}$ is called a *signed measure* if it satisfies

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E_j), \qquad \forall \text{ partitions } \{E_j\} \text{ of } E.$$

Here $\{E_j\}$ is called a (*measurable*) *partition* of E if they are mutually disjoint, measurable and $E = \cup_j E_j$. One immediately deduces that $\mu(\phi) = 0$ and $\mu(X)$ is finite for a signed measure. The latter makes even a non-negative signed measure different from a measure, namely it must be a finite measure.

Given a signed measure on $(X, \mathcal{M})$, its *total variation* (*measure*) is given by

$$|\mu|(E) = \sup\{\sum_j |\mu(E_j)|: \qquad \forall \text{ partitions } \{E_j\} \text{ of } E.$$

Clearly $|\mu|(E_1) \leq |\mu|(E_2)$ if $E_1 \subset E_2$, both in $\mathcal{M}$.

Proposition 5.1. *The total variation of every signed measure on $(X, \mathcal{M})$ is a finite measure on $(X, \mathcal{M})$.*

Proof. We need to show

$$|\mu|(E) = \sum_{j=1}^{\infty} |\mu|(E_j),$$

whenever $\{E_j\}$ is a partition of E.

First we establish subaddivity. Let $\{A_k\}$ be a partition of E. We have

$$\begin{aligned} \sum_k \left|\mu(A_k)\right| &= \sum_k \left|\mu\left(A_k \cap \bigcup_j E_j\right)\right| \\ &\leq \sum_k \sum_j |\mu(A_k \cap E_j)| \\ &= \sum_j \sum_k |\mu(A_k \cap E_j)| \qquad \left(\text{use } \sum_k \sum_j \alpha_{kj} = \sum_j \sum_k \alpha_{kj},\ \forall \alpha_{kj} \geq 0\right) \\ &\leq \sum_j |\mu|(E_j). \qquad \left(\{A_k \cap E_j\}_k \text{ is a partition of } E_j\right) \end{aligned}$$

Taking supremum over all $\{A_k\}$,

$$|\mu|(E) \leq \sum_j |\mu|(E_j).$$

Next, we show the reverse inequality. If $|\mu|(E_j) = \infty$ for some j, then $|\mu|(E) \geq |\mu|(E_j) = \infty$, the inequality holds. Let $|\mu|(E_j) < \infty$ $\forall j$. For every $\varepsilon > 0$, we can find a partition $\left\{E_k^j\right\}$ of E^j for each j such that

$$|\mu|(E_j) \leq \sum_k \left|\mu\left(E_k^j\right)\right| + \frac{\varepsilon}{2^j}.$$

Now, $\{E_k^j\}_{k,j}$ forms a partition of E. We have

$$\begin{aligned} \sum_j |\mu|(E_j) &\leq \sum_j \sum_k \left|\mu\left(E_k^j\right)\right| + \varepsilon \\ &\leq |\mu|(E) + \varepsilon \end{aligned}$$

and we get the desired inequality after letting $\varepsilon \to 0$.

Finally, we show that $|\mu|(X)$ is finite. If on the contrary $|\mu|(X) = \infty$. By the lemma below, we can break X into A_1 and B_1 such that $|\mu(A_1)|, |\mu(B_1)| \geq 1$. As $|\mu|(A_1 + |\mu|(B_1) = |\mu|(X)$, at least one of $\mu(A_1)$ and $\mu(B_1)$ has infinite measure. Assume $|\mu|(A_1) = \infty$. Apply the same argument to A_1 to get A_2, B_2 such that $|\mu(A_2)|, |\mu(B_2)| \geq 1$. Assume again that A_2 has infinite measure. Keep doing this we get $B_{j+1} \subset A_j$, so that $B_j \cap B_k = \phi$, $j \neq k$. Letting $B = \bigcup_j B_j$, we have

$$\mu(B) = \sum_{j=1}^{\infty} \mu(B_j).$$

As $\mu(B)$ is finite, this infinite series converges, and, in particular, $\mu(B_j) \to 0$, but this is in conflict with our construction that $|\mu(B_j)| \geq 1$ for all j. □

Lemma 5.2. *If $|\mu|(E) = \infty$ for some $E \in \mathcal{M}$, then there are disjoint measurable sets A, B in E satisfying $A \cup B = E$ and $|\mu(A)|, |\mu(B)| \geq 1$.*

Proof. For, let $t > 0$, there is some partition $\{E_j\}$ of E such that $\sum_{j=1}^{\infty} |\mu(E_j)| > t$. We fix a large N such that $\sum_{j=1}^{N} |\mu(E_j)| > t$. Rearrange E_j's in the order such that $E_1, E_2, \ldots, E_m$, all $\mu(E_j) < 0$ and $E_{m+1}, \ldots, E_N$, $\mu(E_j) \geq 0$. Then $|\mu(E_1) + \cdots \mu(E_m)| + |\mu(E_{m+1}) + \cdots + \mu(E_N)| > t$, and so, either

$$|\mu(E_1) + \cdots + \mu(E_m)| > \frac{t}{2}$$

or

$$|\mu(E_{m+1}) + \cdots + \mu(E_N)| > \frac{t}{2}.$$

Assume, say, it is the former. We take $A = \bigcup_{j=1}^{m} E_j$. Then $|\mu(A)| > \frac{t}{2}$. Let $B = E \setminus A$. Then

$$\begin{aligned} |\mu(B)| &= |\mu(E) - \mu(A)| \\ &\geq |\mu(A)| - |\mu(E)| \\ &\geq \frac{t}{2} - |\mu(E)| \\ &\geq 1 \end{aligned}$$

if we choose $t > 2$ and $t > 2|\mu(E)| + 2$.

□

The following proposition yields a lot of signed measures from a single measure.

Proposition 5.3. *Let μ be a measure on $(X, \mathcal{M})$ and $f \in L^1(\mu)$. Then*

(a)

$$\lambda(E) = \int_E f d\mu, \quad \forall E \in \mathcal{M}$$

is a signed measure.

(b) Its total variation is given by

$$|\lambda|(E) = \int_E |f| d\mu.$$

Proof. (a) is immediate. To show (b) let $\{E_j\}$ be a measurable partition of E. Then

$$\begin{aligned} \sum_j |\lambda(E_j)| &= \sum_j \left| \int_{E_j} f d\mu \right| \\ &\leq \sum_j \int_{E_j} |f| d\mu \\ &= \int_E |f| d\mu. \end{aligned}$$

Taking supremum over all measurable partitions, we obtain

$$|\lambda|(E) \leq \int_E |f| d\mu.$$

On the other hand, let $A = \{x \in E : f(x) \geq 0\}$ and $B = \{x \in E : f(x) < 0\}$. A and B form a measurable partition of E. By the definition of the total variation, we have

$$|\lambda|(E) \geq |\lambda(A)| + |\lambda(B)| = \int_E |f| d\mu.$$

□

Denote by $M(X)$ or $M(X, \mathcal{M})$ the collection of all signed measures on $(X, \mathcal{M})$. It is a good exercise to verify that $M(X)$ forms a vector space and is complete under the norm

$$\|\mu\| = |\mu| (X).$$

$M(X, \mathcal{M})$ is a Banach space. Given a signed measure μ, let

$$\mu^+ = \frac{1}{2}(|\mu| + \mu), \qquad \mu^- = \frac{1}{2}(|\mu| - \mu).$$

We can write

$$\mu = \mu^+ - \mu^- \text{ and } |\mu| = \mu^+ + \mu^-,$$

where μ^+ and μ^- are finite measures. The decomposition of a signed measure into the difference of two finite measures is called the *Jordan decomposition* of the signed measure. The terminology comes from a corresponding decomposition for functions of bounded variation.

5.2 Radon-Nikodym theorem

Let μ be a measure and λ a measure or signed measure on the same σ-algebra $\mathcal{M}$. The measure λ is called *absolutely continuous* with respect to μ, $\lambda << \mu$ in notation, if every μ-null set is λ-null. λ is *concentrated* on a set $A \in \mathcal{M}$ if $\lambda(E) = \lambda(E \cap A)$, for all $E \in \mathcal{M}$. Note that the set A is not unique. When λ is concentrated on A, it is also concentrated on any set containing A and any set of the form $A \setminus N$ where N is λ-null. We call two measures/signed measures λ_1 and λ_2 *singular* to each other, $\lambda_1 \perp \lambda_2$ in notation, if λ_1 and λ_2 are concentrated respectively on A and B where A and B are disjoint. Clearly, $\lambda_1 \perp \mu$ if λ_1 is concentrated on A and $\mu(A) = 0$.

The following proposition can be derived easily from the definitions above and is left for you to prove.

Proposition 5.4. *Let μ be a measure and λ_i measures or signed measures, $i = 1, 2$. Then*

(a) λ *is concentrated on* $A \Rightarrow |\lambda|$ *is concentrated on* A.

(b) $\lambda_1 \perp \lambda_2 \Rightarrow |\lambda_1| \perp |\lambda_2|$.

(c) $\lambda_1 \perp \mu$, $\lambda_2 \perp \mu \Rightarrow \lambda_1 + \lambda_2 \perp \mu$.

(d) $\lambda_1 << \mu$, $\lambda_2 << \mu \Rightarrow \lambda_1 + \lambda_2 << \mu$.

(e) $\lambda << \mu \Rightarrow |\lambda| << \mu$.

(f) $\lambda_1 << \mu$, $\lambda_2 \perp \mu \Rightarrow \lambda_1 \perp \lambda_2$.

(g) $\lambda << \mu$, $\lambda \perp \mu \Rightarrow \lambda = 0$.

Example 5.1. Consider $(\mathbb{R}, \mathcal{M})$ where $\mathcal{M}$ is the σ-algebra consisting of all Lebesgue measurable sets. For $f \in L^1(\mathbb{R})$, define

$$\nu(E) = \int_E f \, d\mathcal{L}^1.$$

Then $\nu << \mathcal{L}^1$. On the other hand, consider

$$\lambda = \sum_{j=1}^{N} \alpha_j \delta_{x_j}$$

where $\{x_1, \ldots, x_N\} \subset \mathbb{R}$ and $\alpha_j \in \mathbb{R}$. Then λ is a signed measure on $(\mathbb{R}, \mathcal{M})$ (in fact, on $(\mathbb{R}, \mathcal{P}_{\mathbb{R}})$.) As $\mathcal{L}^1$ is concentrated on $\mathbb{R}$, so does it on $\mathbb{R} \setminus \{x_1, \ldots, x_N\}$. As λ is concentrated on $\{x_1, \ldots, x_N\}$, λ and $\mathcal{L}^1$ are singular. By Proposition 5.3, all ν's are singular to λ too.

Absolute continuity and mutual singularity describe two extreme relations between two measures. It is striking that they are sufficient in certain sense for the description of the relation between two measures. This result is contained in the following two theorems.

Theorem 5.5 (**Lebesgue Decomposition**). *Let μ be a σ-finite measure and λ a signed measure on $(X, \mathcal{M})$. There exist a unique pair of signed measures $(\lambda_{ac}, \lambda_s)$, $\lambda_{ac} << \mu$ and $\lambda_s \perp \mu$, such that $\lambda = \lambda_{ac} + \lambda_s$.*

Theorem 5.6 (**Radon-Nikodym Theorem**). *Let μ be a σ-finite measure and λ a signed measure on $(X, \mathcal{M})$ such that $\lambda << \mu$. There exists a unique $h \in L^1(\mu)$ such that*

$$\lambda(E) = \int_E h \, d\mu, \qquad \forall E \in \mathcal{M}.$$

The function h is called the *Radon-Nikodym derivative* of λ with respect to μ and will be denoted by $d\lambda/d\mu$.

We will prove these two theorems in one stroke. The proof is due to von Neumann.

Proof. Step 1. Assume that both μ and λ are finite measures. Setting $\rho = \mu + \lambda$ and define a functional on $L^2(\rho)$ by

$$\Lambda\varphi = \int \varphi \, d\lambda, \quad \varphi \in L^2(\rho).$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} |\Lambda\varphi| &\leq \int |\varphi| \, d\lambda \\ &\leq \int |\varphi| \, d\rho \\ &\leq \sqrt{\rho(X)} \, \|\varphi\|_{L^2(\rho)} \, , \end{aligned}$$

so Λ is a bounded linear functional on $L^2(\rho)$. By the self-duality of $L^2(\rho)$, there exists some $g \in L^2(\rho)$ such that

$$\int \varphi \, d\lambda = \int \varphi g \, d\rho, \qquad \forall \varphi \in L^2(\rho). \tag{5.1}$$

Taking $\varphi = \chi_E, E \in \mathcal{M}$, φ belongs to $L^2(\rho)$ and we have

$$\lambda(E) = \int_E g \, d\rho,$$

and

$$\frac{1}{\rho(E)} \int_E g \, d\rho = \frac{\lambda(E)}{\rho(E)} \in [0, 1].$$

By a previous exercise, $g \in [0, 1]$ ρ-a.e. By redefining g in a null set, we may assume $g(X) \subset [0, 1]$. Let

$$\lambda_{ac}(E) = \lambda(E \cap A),$$

and

$$\lambda_s(E) = \lambda(E \cap B),$$

where $A = \{x \in X : g(x) \in [0, 1)\}$ and $B = \{x \in X : g(x) = 1\}$. We have $\lambda = \lambda_{ac} + \lambda_s$.

We rewrite (5.1) as

$$\int \varphi(1 - g) \, d\lambda = \int \varphi g \, d\mu. \tag{5.2}$$

Taking $\varphi = \chi_B$, we get

$$0 = \int_B d\mu = \mu(B).$$

We conclude that $\lambda_s \perp \mu$, since by definition λ_s is concentrated on B. Taking $\varphi = \chi_E(1 + g + \cdots + g^n)$ in (5.2), we have

$$\int \chi_E(1 - g^{n+1}) \, d\lambda = \int \chi_E(1 + g + \cdots + g^n) g \, d\mu.$$

The left hand side of this relation satisfies

$$\begin{aligned} & \int \chi_E(1 - g^{n+1}) \, d\lambda \\ = & \int_E (1 - g^{n+1}) \, d\lambda \\ = & \int_{E \cap A} (1 - g^{n+1}) \, d\lambda \\ \to & \, \lambda(E \cap A) = \lambda_{ac}(E), \qquad \text{as } n \to \infty, \end{aligned}$$

where the monotone convergence theorem has been used in the last step. On the other hand, the right hand side becomes

$$\begin{aligned}\int_E (1+g+\cdots+g^n)g\,d\mu &= \int_{E\cap A} g\frac{1-g^{n+1}}{1-g}\,d\mu \\ &\to \int_{E\cap A}\frac{g}{1-g}\,d\mu \qquad \text{as } n\to\infty.\end{aligned}$$

Setting $h = g/(1-g)$, h is non-negative and belongs to $\in L^1(\mu)$. We conclude that

$$\lambda_{ac}(E) = \int_E h\,d\mu, \quad \forall E \in \mathcal{M},$$

so, $\lambda_{ac} << \mu$.

Step 2. Consider the case that μ is a finite measure and λ is a signed measure. Applying what has been proved to λ^+ and λ^-, the Jordan decomposition of λ, we have

$$\begin{aligned}\lambda^+ = \lambda^+_{ac} + \lambda^+_s, &\qquad \lambda^- = \lambda^-_{ac} + \lambda^-_s, \\ \lambda^+_{ac}, \quad \lambda^-_{ac} << \mu, &\qquad \lambda^+_s, \quad \lambda^-_s \perp \mu.\end{aligned}$$

By Proposition 5.4, $\lambda = \lambda_{ac} + \lambda_s$, $\lambda_{ac} \equiv \lambda^+_{ac} - \lambda^-_{ac}$, $\lambda_s = \lambda^+_s - \lambda^-_s$, and $\lambda_{ac} << \mu$ and $\lambda_s \perp \lambda_{ac}$. Moreover, if

$$\lambda^{\pm}_{ac} = \int h^{\pm}\,d\mu,$$

$$\lambda_{ac} = \int h\,d\mu, \qquad \text{where } h \equiv h^+ - h^- \in L^1(\mu).$$

Step 3. Let μ be a σ-finite measure and λ a signed measure. Let $\{X_j\}$ be a measurable partition of X with finite measure. Let $\mu_j = \mu\lfloor_{X_j}$ and $\lambda_j = \lambda\lfloor_{X_j}$. By the previous step,

$$\lambda_j = \lambda^j_{ac} + \lambda^j_s, \qquad \lambda^j_{ac} << \mu_j, \quad \lambda^j_s \perp \mu_j,$$

$$\lambda^j_{ac}(E) = \int_E h_j\,d\mu_j = \int_E h_j\chi_{X_j}\,d\mu, \quad h_j \in L^1(\mu_j), h_j\chi_{X_j} \in L^1(\mu).$$

Letting $\lambda_{ac} = \sum_{j=1}^{\infty}\lambda^j_{ac}$ and $\lambda_s = \sum_{j=1}^{\infty}\lambda^j_s$, we have

$$\lambda = \lambda_{ac} + \lambda_s, \quad \lambda_{ac} << \mu, \quad \lambda_s \perp \mu,$$

and

$$\lambda_{ac}(E) = \int_E h\, d\mu, \quad \text{where } h = \sum_{j=1}^{\infty} h_j \chi_{X_j}.$$

By Proposition 5.3,

$$\infty > |\lambda|(X) = \int_X |h| d\mu,$$

so $h \in L^1(\mu)$.

Step 4. To finish the proof we establish uniqueness. Suppose there is a pair (λ_1, λ_2) with $\lambda_1 << \mu$ and $\lambda_2 \perp \mu$ such that

$$\lambda = \lambda_1 + \lambda_2.$$

As $\lambda = \lambda_{ac} + \lambda_s$, $\lambda_1 + \lambda_2 = \lambda_{ac} + \lambda_s$, it implies that $\lambda_1 - \lambda_{ac} = \lambda_s - \lambda_2$. As $\lambda_1 - \lambda_{ac} << \mu$ and $\lambda_s - \lambda_2 \perp \mu$, by Proposition 5.3, $\lambda_1 - \lambda_{ac} = \lambda_s - \lambda_2 = 0$, that is, $\lambda_1 = \lambda_{ac}$ and $\lambda_2 = \lambda_s$. $\square$

We deduce a significant result from the Radon-Nikodym theorem.

Let μ be a signed measure on $(X, \mathcal{M})$. From the relation $|\mu(E)| \leq |\mu|(E)$, $\forall E \in \mathcal{M}$, we know that $\mu << |\mu|$. By Radon-Nikodym theorem, we can find some $h \in L^1(|\mu|)$ such that

$$\mu(E) = \int_E h\, d|\mu|.$$

From

$$\frac{1}{|\mu|(E)} \int_E h\, d|\mu| = \frac{\mu(E)}{|\mu|(E)} \in [-1, 1],$$

we know that $|h| \leq 1$ a.e. We claim that in fact $|h| = 1$ a.e. For, let

$$A_r = \{x \in X : |h|(x) < r\}, \quad r \in (0, 1).$$

For any partition $\{A_j\}$ of A_r,

$$\begin{aligned}
\sum_j |\mu(A_j)| &= \sum_j \left| \int_{A_j} h\, d|\mu| \right| \\
&\leq \sum_j \int_{A_j} |h|\, d|\mu| \\
&\leq r \sum_j |\mu|(A_j) \\
&= r\, |\mu|(A_r).
\end{aligned}$$

Taking supremum over all partitions $\{A_j\}$,

$$|\mu|(A_r) \leq r\,|\mu|(A_r).$$

As $|\mu|(A_r) \leq |\mu|(X) < \infty$, this forces A_r has $|\mu|$-measure zero. So, $|h| = 1$ $|\mu|$-a.e. After redefining h on a set of measure zero, we may assume $|h| = 1$ everywhere. Thus we have

Proposition 5.7. *Let μ be a signed measure on $(X, \mathcal{M})$.*

(a) There exists an $h \in L^1(|\mu|)$, $|h| \equiv 1$, such that $d\mu = h\,d|\mu|$, that is,

$$\mu(E) = \int_E h\,d|\mu|\,, \quad \forall E \in \mathcal{M}.$$

(b) There are disjoint measurable sets A and B such that

$$\begin{aligned}\mu^+(E) &= \mu(E \cap A)\\ \mu^-(E) &= -\mu(E \cap B), \quad \forall E \in \mathcal{M}.\end{aligned}$$

(c) If $\mu = \lambda_1 - \lambda_2$ where λ_i are measures, then $\lambda_1 \geq \mu^+$ and $\lambda_2 \geq \mu^-$.

Proof. (a) Already done.
(b) Let $A = \{x \in X : h(x) = 1\}$ and $B = \{x \in X : h(x) = -1\}$. Then $A \dot\cup B = X$ and

$$\begin{aligned}\mu^+(E) &= \frac{1}{2}(|\mu|(E) + \mu(E))\\ &= \frac{1}{2}\int_E (1+h)\,d|\mu|\\ &= \int_{E\cap A} d|\mu| \quad (\text{use } 1+h = 2 \text{ on } A,\ 1+h = 0 \text{ on } B)\\ &= \int_{E\cap A} d\mu\\ &= \mu(E \cap A).\end{aligned}$$

Similarly,

$$
\begin{aligned}
\mu^-(E) &= \frac{1}{2}(|\mu|(E) - \mu(E)) \\
&= \frac{1}{2}\int_E (1-h)\, d|\mu| \\
&= \int_{E\cap B} d|\mu| \\
&= -\int_{E\cap B} d\mu \\
&= -\mu(E\cap B).
\end{aligned}
$$

(c) As $\mu = \lambda_1 - \lambda_2 \le \lambda_1$,

$$\mu^+(E) = \mu(E\cap A) \le \lambda_1(E\cap A) \le \lambda_1(E).$$

As $-\mu = \lambda_2 - \lambda_1 \le \lambda_2$,

$$\mu^-(E) = -\mu(E\cap B) \le \lambda_2(E\cap B) \le \lambda_2(E).$$

□

The existence of disjoint A and B such that $\mu^+ = \mu\lfloor_A$ and $\mu^- = \mu\lfloor_B$ is called the *Hahn decomposition* of μ. Proposition 5.6(c) shows certain minimal property of the decomposition.

5.3 The Dual Space of $C_0(X)$

The space $C_c(X)$ may not be complete under the supnorm. For instance, the function e^{-x^2} is the uniform limit of a sequence of functions in $C_c(\mathbb{R})$, but it is positive everywhere. The space of all bounded, continuous functions on a topological space forms a Banach space where $C_c(X)$ is its subspace. We denote the closure of $C_c(X)$ in in this space by $C_0(X)$. It is a Banach space. In fact, when X is a locally compact Hausdorff space, one can show that it consists of all continuous functions that vanish at infinity. More precisely, f belongs to $C_0(X)$ if and only if, for every $\varepsilon > 0$, there is a compact set K such that $|f|$ is less than ε outside K. Of course, $C_0(X)$ is equal to $C_c(X)$ when X is compact. As $C_c(X)$ is dense in $C_0(X)$, the dual space of $C_c(X)$ can be identified with the dual space of $C_0(X)$.

From now on we take X to be a locally compact Hausdorff topological space. A signed measure $\mu \in M(X) = M(X, \mathcal{B})$ is called *regular* if every $E \in \mathcal{B}$ is inner and outer regular, i.e.,

$$\begin{aligned}|\mu|(E) &= \inf\{|\mu|(G) : E\subset G,\ G \text{ open}\}, \quad \text{and}\\ &= \sup\{|\mu|(K) : K\subset E,\ K \text{ compact}\}.\end{aligned}$$

We use $M_r(X)$ to denote all regular Borel signed measures on $(X, \mathcal{B})$.

Proposition 5.8. *(a) $M_r(X)$ forms a closed subspace of $M(X)$ under the norm $\|\mu\| = |\mu|(X)$,*

(b) $\mu = \mu^+ - \mu^-$, $\mu \in M_r(X)$ if and only if $\mu^\pm \in M_r(X)$,

(c) For $\mu \in M_r(X)$ and $f \in L^1(|\mu|)$, the measure λ given by

$$\lambda(E) = \int_E f\, d|\mu|,$$

belongs to $M_r(X)$.

I leave the proof of this proposition as an exercise.

Each regular Borel signed measure induces a bounded linear functional on $C_c(X)$. In fact, we used to define

$$\Lambda\varphi = \int \varphi\, d\mu, \quad \forall \varphi \in C_c(X),$$

when μ is a measure. Now, for a signed measure, we could use Proposition 5.7(a) to define

$$\Lambda\varphi = \int \varphi h\, d|\mu| \ ,$$

or,

$$\Lambda\varphi = \int \varphi\, d\mu^+ - \int \varphi\, d\mu^-.$$

With this definition,

$$|\Lambda\varphi| \le \int |\varphi h|\ d|\mu| \le |\mu|(X)\, \|\varphi\|_\infty,$$

so

$$\|\Lambda\| \le |\mu|(X) = \|\mu\|, \tag{5.3}$$

holds and, in particular, it shows that $\Lambda \in C_c(X)'$.

Consider the map $\Phi : M_r(X) \to C_c(X)'$ by $\Phi : \mu \mapsto \Lambda$. We claim that that Φ is linear.

Indeed, writing $\int \varphi\, d\mu \equiv \int \varphi h\, d|\mu|$, we first prove that

$$\int \varphi\, d(\mu_1 + \mu_2) = \int \varphi\, d\mu_1 + \int \varphi\, d\mu_2, \quad \forall \mu_1, \mu_2 \in M_r(X)$$

and

$$\int \varphi \, d(c\mu) = c \int \varphi \, d\mu, \quad \forall c \in \mathbb{R}.$$

The second one follows directly from the definition. For the first one, we have

$$\int \varphi \, d(\mu_1 + \mu_2) = \int \varphi \, d(\mu_1 + \mu_2)^+ - \int \varphi \, d(\mu_1 + \mu_2)^- \ ,$$

$$\int \varphi \, d\mu_1 = \int \varphi \, d\mu_1^+ - \int \varphi \, d\mu_1^- \ ,$$

and

$$\int \varphi \, d\mu_2 = \int \varphi \, d\mu_2^+ - \int \varphi \, d\mu_2^- \ .$$

It suffices to show that

$$\int \varphi \, d(\mu_1 + \mu_2)^+ + \int \varphi \, d\mu_1^- + \int \varphi \, d\mu^- = \int \varphi \, d(\mu_1 + \mu_2)^- + \int \varphi \, d\mu_1^+ + \int \varphi \, d\mu_2^+. \tag{5.4}$$

Observe that for any two finite measures λ and μ on the same σ-algebra,

$$\int f \, d(\lambda + \mu) = \int f \, d\lambda + \int f \, d\mu, \qquad \forall \text{ bounded measurable} f,$$

As

$$\mu_1 + \mu_2 = (\mu_1 + \mu_2)^+ - (\mu_1 + \mu_2)^- \ ,$$

$$\mu_1 = \mu_1^+ - \mu_1^-, \text{ and } \qquad \mu_2 = \mu_2^+ - \mu_2^-,$$

we have

$$(\mu_1 + \mu_2)^+ + \mu_1^- + \mu_2^- = (\mu_1 + \mu_2)^- + \mu_1^+ + \mu_2^+ \ .$$

Therefore,

$$\int \varphi \, d(\mu_1 + \mu_2)^+ + \int \varphi \, d\mu_1^- + \int \varphi \, d\mu^- = \int \varphi \, d(\mu_1 + \mu_2)^- + \int \varphi \, d\mu_1^+ + \int \varphi \, d\mu_2^+,$$

and (5.4) follows.

Finally, let us show that the map from μ to Λ is norm-preserving. In view of (5.3), it remains to show $\|\Lambda\| \geq \|\mu\|$. Indeed, from

$$\Lambda\varphi = \int \varphi h d\mu$$

we choose a suitable φ to approximate h. Recall that $|h| \equiv 1$ and $|\mu|$ is a finite measure, by Lusin's theorem (Theorem 2.12), for each $\varepsilon > 0$, there exists some $\varphi \in C_c(X)$ satisfying (a) $|\varphi| \leq |h| = 1$ and (b) $|\mu|(A) < \varepsilon$ where $A = \{x : \ \varphi(x) \neq$

$h(x)\}$. We have

$$
\begin{aligned}
\|\Lambda\| &\geq |\Lambda\varphi| \\
&= \left|\int \varphi h d|\mu|\right| \\
&\geq |\mu|(X \setminus A) - \varepsilon \\
&\geq |\mu|(X) - 2\varepsilon.
\end{aligned}
$$

The desired inequality follows by letting $\varepsilon \to 0$.

Summarizing, we have shown that the map $\mu \mapsto \Lambda$ is a norm-preserving, linear map from $M_r(X)$ to $C_0(X)'$.

Theorem 5.9. *Let X be a locally compact Hausdorff space. The $\Phi : M_r(X, \mathcal{B}) \to C_0(X)'$ is a norm-preserving, bijective linear map. In other words, the dual space of $C_0(X)$ is equal to $M_r(X, \mathcal{B})$.*

This is the full version of Riesz representation theorem. It is different from Theorem 2.8 for two points. First, it deals only with bounded linear functionals while Theorem 2.8 deals with positive linear functionals which may not be bounded. Second, since we are only concerned with the dual of $C_c(X)$ only, we do not consider outer measures here.

Lemma 5.10. *Setting as above, there exists a positive linear functional Λ_0 on $C_c(X)$ satisfying*

$$|\Lambda f| \leq \Lambda_0(|f|) \leq \|\Lambda\| \|f\|_\infty, \quad \forall f \in C_c(X).$$

Proof. For $f \in C_c^+(X)$, we define

$$\Lambda_0 f = \sup \{|\Lambda\varphi| : |\varphi| \leq f, \ \varphi \in C_c(X)\}.$$

Clearly, $\Lambda_0 f \geq 0$, $\Lambda_0 f_2 \geq \Lambda_0 f_1$ if $f_2 \geq f_1 \geq 0$ and $\Lambda_0(cf) = c\Lambda_0 f$. We claim

$$\Lambda_0(f_1 + f_2) = \Lambda_0 f_1 + \Lambda_0 f_2, \quad f_1, f_2 \in C_c^+(X).$$

for, observe that

$$|\Lambda\varphi| \leq \|\Lambda\| \|\varphi\|_\infty \leq \|\Lambda\| \|f\|_\infty, \quad \text{if } |\varphi| \leq f.$$

Taking supremum over all these φ,

$$|\Lambda_0 f| \leq \|\Lambda\| \|f\|_\infty. \tag{5.5}$$

In particular, $\Lambda_0 f$ is finite. Let $f_1, f_2 \in C_c^+(X)$. For $\varepsilon > 0$, there are φ_1, φ_2,

$|\varphi_1| \le f_1$, $|\varphi_2| \le f_2$, such that

$$\Lambda_0 f_1 \le |\Lambda\varphi_1| + \varepsilon, \qquad \Lambda_0 f_2 \le |\Lambda\varphi_2| + \varepsilon.$$

Pick $\sigma_1, \sigma_2 \in \{1, -1\}$ such that $|\Lambda\varphi_1| = \sigma_1\Lambda\varphi_1$, $|\Lambda\varphi_2| = \sigma_2\Lambda\varphi_2$, then

$$\begin{aligned}\Lambda_0 f_1 + \Lambda_0 f_2 &\le \Lambda(\sigma_1\varphi_1 + \sigma_2\varphi_2) + 2\varepsilon \\ &\le \Lambda_0(f_1 + f_2) + 2\varepsilon.\end{aligned}$$

So,

$$\Lambda_0 f_1 + \Lambda_0 f_2 \le \Lambda_0(f_1 + f_2).$$

To get the reverse inequality, let $\varphi \in C_c(X)$, $|\varphi| \le f_1 + f_2$ and set

$$\varphi_1 = \frac{f_1\varphi}{f_1 + f_2}, \quad \varphi_2 = \frac{f_2\varphi}{f_1 + f_2} \quad \text{on } V,$$

and and $\varphi_1 = \varphi_2 = 0$ on $X \setminus V$ where $V = \{x \in X : (f_1 + f_2)(x) > 0\}$. Note that $\varphi_1, \varphi_2 \in C_c(X)$,$|\varphi_1| \le f_1$, $|\varphi_2| \le f_2$, and $\varphi = \varphi_1 + \varphi_2$. We have

$$\begin{aligned}|\Lambda\varphi| &= |\Lambda\varphi_1 + \Lambda\varphi_2| \\ &\le |\Lambda\varphi_1| + |\Lambda\varphi_2| \\ &\le \Lambda_0 f_1 + \Lambda_0 f_2.\end{aligned}$$

Taking supremum over all these φ,

$$\Lambda_0(f_1 + f_2) \le \Lambda_0 f_1 + \Lambda_0 f_2.$$

For $f \in C_c(X)$, we set

$$\Lambda_0 f = \Lambda_0 f^+ - \Lambda_0 f^-.$$

Using the old trick

$$(f_1 + f_2)^+ - (f_1 + f_2)^- = f_1 + f_2 = f_1^+ - f_1^- + f_2^+ - f_2^-,$$

it is routine to check that Λ_0 is our desired positive linear functional on $C_c(X)$. $\square$

Now, we can prove Theorem 5.9. To show that Φ is onto, we need to find some μ that satisfies $\Phi(\mu) = \Lambda$ for any given Λ. Indeed, applying Theorem 2.8 to Λ_0, we can find a Radon measure λ such that

$$\Lambda_0 f = \int f \, d\lambda, \quad \forall f \in C_c(X).$$

Here λ satisfies

- $\lambda(E) = \inf\{\lambda(G) : E \subset G, G \text{ open}\}, \qquad \forall E \subset X,$
- $\lambda(E) = \sup\{\lambda(K) : K \subset E, K \text{ compact}\}$ if $\lambda(E) < \infty, \qquad \forall E \in \mathcal{B}.$

By Lemma 5.10,

$$\left|\int f \, d\lambda\right| = |\Lambda_0 f| \leq \|\Lambda\| \, \|f\|_\infty, \quad \forall f \in C_c(X).$$

From the proof of Theorem 2.8,

$$\lambda(X) = \sup\{|\Lambda_0 f| : 0 \leq f \leq 1 \text{ on } X\}.$$

In view of this, we have

$$\lambda(X) \leq \|\Lambda\|,$$

i.e., λ belongs to $M_r(X)$.

From the definition of Λ_0 we also have

$$\begin{aligned} |\Lambda f| &\leq \Lambda_0 f^+ + \Lambda_0 f^- \\ &= \|f\|_{L^1(\lambda)}, \qquad \forall f \in C_c(X). \end{aligned}$$

As $C_c(X)$ is dense in $L^1(\lambda)$, Λ can be extended to a bounded linear functional on $L^1(\lambda)$. Since λ is a finite measure, by L^1-L^∞ duality there exists some $h \in L^\infty(\lambda)$ such that

$$\Lambda f = \int f h \, d\lambda \quad \forall f \in L^1(\lambda),$$

and the operator norm of Λ as a linear functional on $L^1(\lambda)$ is equal to $\|h\|_\infty$. Using $|\Lambda f| \leq \|f\|_{L^1(\lambda)}$, we have $\|h\|_\infty \leq 1$. On the other hand, we have

$$\begin{aligned} \|\Lambda\| &= \sup\{|\Lambda\varphi| : \ \varphi \in C_c(X), |\varphi| \leq 1\} \\ &\leq \int |h| d\lambda \\ &\leq \lambda(X) \\ &\leq \|\Lambda\|, \end{aligned}$$

which forces $|h| = 1$ λ-a.e. and $\lambda(X) = \|\Lambda\|$. Now we set

$$\mu(E) = \int_E h d\lambda, \quad \forall E \in \mathcal{B}.$$

By Proposition 5.8, $\mu \in M_r(X)$. By Proposition 5.3, $|\mu| = \lambda$ and so $\|\mu\| = \|\Lambda\|$. We conclude that Φ is a norm-preserving linear bijection from $M_r(X, \mathcal{B})$ to $C_0(X)'$. The proof of Theorem 5.9 is completed.

5.4 Weak* Convergence of Measures

Let E be a normed space and E' its dual space. A sequence $\{\Lambda_k\} \subset E'$ is called *weakly* convergent* to $\Lambda \in E'$ if

$$\Lambda_k x \to \Lambda x, \quad \forall x \in E,$$

as $k \to \infty$. Consider $L^p(\mu)$, $1 < p < \infty$, we know that $L^p(\mu) = L^q(\mu)'$ where q is conjugate to p. For $f_k \in L^p$, $f_k \overset{*}{\rightharpoonup} f$ if and only if

$$\int f_k g \, d\mu \to \int f g \, d\mu, \quad \forall g \in L^q(\mu).$$

This turns out to be the same as $f_n \rightharpoonup f$. So when $1 < p < \infty$, weak* and weak convergence are the same, depending on whether to regard $L^p(\mu)$ as the "base" space or the dual of $L^q(\mu)$.

Applying to the case the space $C_0(X)$ where X is a locally compact Hausdorff space, by the representation theorem

$$\mu_n \overset{*}{\rightharpoonup} \mu \quad \text{in } M_r(X)$$

if and only if

$$\int \varphi \, d\mu_n \to \int \varphi \, d\mu, \quad \forall \varphi \in C_c(X).$$

The following result is sometimes called Helly selection theorem.

Theorem 5.11. *Let E be a separable normed space. Every bounded sequence in E' contains a weakly* convergent subsequence.*

This theorem can be proved as Theorem 5.8 and we omitted its proof. As a special case we have

Corollary 5.12. *Let $C_0(X)$ be separable. Every sequence $\{\mu_k\}$ in $M_r(X)$, $\|\mu_k\| \le M$ for some M, contains a subsequence μ_{k_j} which $\mu_{k_j} \overset{*}{\rightharpoonup} \mu$ for some $\mu \in M_r(X)$.*

One can show that the space $C_0(X)$ is separable when X is a compact metric space or it is an open set in $\mathbb{R}^n$.

Recall that even $\|f_k\|_{L^1}$ is uniformly bounded, we cannot always pick a weakly convergent subsequence f_{n_j} in $L^1(\mu)$. As a typical example we may take $\{f_k\}$ satisfying $\text{spt} f_k = [a_k, b_k]$ shrinks to $\{x_0\}$ and $\|f_k\|_{L^1} = 1$ for all k. Then no subsequence of f_k converges weakly. However, if we regard an L^1-function as a measure by setting $d\mu_k = f_k \, d\mathcal{L}^1$, μ_k now belongs to the larger space $M_r(\mathbb{R})$ and $\|\mu_k\| = \|f_k\|_{L^1} = 1$. By Corollary 5.10, it has weak* subconvergence $\mu_{k_j} \overset{*}{\rightharpoonup} \mu$. In fact, it is clear that the entire sequence converges weakly* to the Dirac measure δ_{x_0}. The lesson is, by enlarging the space from $L^1(\mathbb{R})$ to $M_r(\mathbb{R})$, we get subconvergence.

As an application of this property, we prove

Theorem 5.13. *Consider $L^1(\Omega)$ where Ω be a bounded, open set in $\mathbb{R}^n$. Let $\{f_k\}$ be a bounded sequence in $L^1(\Omega)$ which is uniformly integrable. Then $\{f_k\}$ contains a weakly convergent subsequence in $L^1(\Omega)$.*

In fact, the converse is true, namely, if $f_k \overset{*}{\rightharpoonup} f$ for some $f \in L^1(\Omega)$, then $\{f_k\}$ is bounded in $L^1(\Omega)$ and uniformly integrable. This necessary and sufficient condition for the weak subconvergence of an L^1-sequence is called Dunford-Pettis theorem.

Proof of Theorem 5.13. It suffices to consider the case $f_k \geq 0$. Let $d\mu_n = f_n \, d\mathcal{L}^n$. Then $\|\mu_k\| = \|f_k\|_{L^1}$ is uniformly bounded by some constant M. By Corollary 5.10, there exist $\{\mu_{k_j}\}$ and $\mu \in M_r(\Omega)$ such that

$$\int \varphi \, d\mu_{k_j} \to \int \varphi \, d\mu, \quad \forall \varphi \in C_c(\Omega) \qquad \text{as } j \to \infty. \tag{5.6}$$

For simplicity, we assume the entire sequence converges weakly*. By Lebesgue decomposition (with respect to $\mathcal{L}^n$),

$$\mu = f\mathcal{L}^n + \nu, \quad f \in L^1(\Omega), \ \nu \perp \mathcal{L}^n.$$

It suffices to show that $\nu \equiv 0$ so that

$$\int \varphi f_k \, dx \to \int \varphi f \, dx, \quad \forall \varphi \in C_c(\overline{\Omega}) = C(\overline{\Omega}). \tag{5.7}$$

Suppose on the contrary that there is some $A \in \mathcal{B}$, $\nu(A) > 0$, such that $\mathcal{L}^n(A) = 0$. Letting $a_0 \equiv \nu(A) = \mu(A)$, by the regularity of the Lebsegue measure and μ, for $\delta > 0$, there is an open G containing A such that

$$\mu(G) < \delta,$$

and there is a compact set $K \subset A$ such that

$$\mathcal{L}^n(K) \geq \frac{a_0}{2}.$$

For $\varepsilon < a_0/2$, by uniform integrability, there is a $\delta_1 > 0$ such that

$$\int_E f_k \, dx < \varepsilon, \quad \forall E \in \mathcal{B}, \ \mathcal{L}^n(E) < \delta_1, \ \forall k \geq 1.$$

We take $\delta = \delta_1$ and pick $\varphi \in C_c(\Omega)$ such that $\varphi \equiv 1$ on K, $0 \leq \varphi \leq 1$, $\text{spt}\varphi \subset G$. Then

$$\int_G \varphi f_k \, dx \leq \int_G f_k \, dx < \varepsilon.$$

Letting $k \to \infty$, by (5.6)

$$\int_G \varphi \, d\mu \le \varepsilon.$$

However, as $\varphi \equiv 1$ on K, we get

$$\mu(K) \le \int_G \varphi \, d\mu \le \varepsilon,$$

contradicting our choice of ε. So $\nu \equiv 0$.

We still have to show (5.7) holds for all $\varphi \in L^\infty(\Omega) = L^1(\Omega)'$. First, we claim that it holds for all $\varphi = \chi_G$, where G is open. For, let $\{G_j\} \uparrow G, G_j \subset\subset G_{j+1}$ and φ_j, $\overline{G_j} < \varphi_j < G_{j+1}$. The claim follows from the uniform integrability of $\{f_k\}$. Next, using outer regularity and uniform integrability we know that (5.7) holds for $\varphi = \chi_E$ for $E \in \mathcal{B}$. Consequently, it also holds for all simple functions s. Now, let $\varphi \in L^\infty(\Omega)$, say, $|\varphi| \le M$. For $\varepsilon > 0$, let $-M-1 = a_1 < a_2 < \cdots < a_N = M+1$, $\Delta a_j < \varepsilon$. Define

$$s = \sum_{j=1}^{N-1} a_j \chi_{A_j}, \quad A_j = \{x \in \Omega : a_j \le \varphi(x) < a_{j+1}\}.$$

Then $\|s - \varphi\|_\infty < \varepsilon$, so

$$\begin{aligned} \left| \int \varphi f_k - \int \varphi f \right| &\le \left| \int (\varphi - s)(f_k - f) \right| + \left| \int s(f_k - f) \right| \\ &\le \varepsilon \times 2M + \left| \int s(f_k - f) \right|. \end{aligned}$$

Letting $n \to \infty$,

$$\overline{\lim_{k\to\infty}} \left| \int \varphi f_k - \int \varphi f \right| \le 2M\varepsilon,$$

and we finally conclude that (5.7) holds for all φ in $L^\infty(\Omega)$. $\square$

5.5 Herglotz-Riesz Theorem

We present another application of the Riesz representation theorem.

Recall that a harmonic function is the real part of an analytic function and it satisfies

$$u_{xx} + u_{yy} = 0$$

in the plane. Let $D_R = \{z : |z| < R\}$, $z = x + iy$, be the disk of radius R. It is well-known that given a continuous function g on the boundary $|z| = R$, $g(\theta)$, $z = Re^{i\theta}$, $\theta \in [0, 2\pi]$, there is a unique harmonic function u in D_R which is equal to g on its boundary. In fact, this harmonic function is given by the Poisson formula

$$u(z) = \frac{1}{2\pi} \int_{|w|=R} \frac{R^2 - |z|^2}{|w - z|^2} g(\theta)\, d\theta. \tag{5.8}$$

The question is: Instead of continuous data g, what are the most general data that can be imposed on the boundary of D_R to get solvability?

It turns out we have

Theorem 5.14. *Let u be a non-negative harmonic function defined in $D_1 = \{z : |z| < 1\}$. There exists a unique Radon measure μ on S^1 such that*

$$\int u(re^{i\theta})\varphi(\theta)\, dx \to \int \varphi(\theta)\, d\mu(\theta), \quad \forall \varphi \in C(S^1), \tag{5.9}$$

as $r \uparrow 1$. In fact,

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\alpha - \theta) + r^2}\, d\mu(\theta), \;\; z = re^{i\alpha}, \;\; r \in (0,1), \;\; \alpha \in [0, 2\pi]. \tag{5.10}$$

Thus every non-negative harmonic function in D_1 has a boundary value which is a Radon measure. Moreover, every Radon measure on the boundary generates a harmonic function by the formula (5.10).

Proof. Let $C(S^1)$ be the space consists of all 2π–periodic, continuous functions. For $0 < r < 1$, set

$$\Lambda_r \varphi = \int_0^{2\pi} u(re^{i\theta})\varphi(\theta)\, d\theta.$$

We have

$$\begin{aligned} |\Lambda_r \varphi| &\le \int_0^{2\pi} u(re^{i\theta})\, |\varphi(\theta)|\, d\theta \\ &\le \|\varphi\|_\infty \int_0^{2\pi} u(re^{i\theta})\, d\theta. \end{aligned}$$

Setting $z = 0$ in (5.8) where now $R = r$ and $g = u(re^{i\theta})$, we obtain the mean value property of the harmonic function

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})\, d\theta.$$

Therefore,

$$|\Lambda_r \varphi| \le 2\pi u(0)\, \|\varphi\|_\infty\,,$$

and

$$\|\Lambda_r\| \le 2\pi u(0) \equiv M < \infty.$$

By Corollary 5.10 or Theorem 2.8, there exist $\{r_j\} \uparrow 1$ and a regular Borel measure μ such that

$$\Lambda_{r_j}\varphi = \int_0^{2\pi} u(r_j e^{i\theta})\varphi(\theta)\, d\theta \to \int_0^{2\pi} \varphi(\theta)\, d\mu(\theta), \quad r_j \uparrow 1.$$

We will later improve the convergence from a sequence $r_j \uparrow 1$ to for all $r \uparrow 1$.

Now, for a fixed $z = re^{i\alpha}$, $0 < r < 1$, the function

$$\varphi_R(\theta) = \frac{R^2 - r^2}{|Re^{i\theta} - re^{i\alpha}|^2} = \frac{R^2 - r^2}{1 - 2r\cos(\alpha - \theta) + r^2}$$

converges uniformly to

$$\varphi_1(\theta) = \frac{1 - r^2}{1 - 2r\cos(\alpha - \theta) + r^2}$$

as $R \uparrow 1$. It follows that

$$\Lambda_{r_j}\varphi_{r_j} \to \int_0^{2\pi} \varphi_1(\theta) d\mu(\theta), \qquad r_j \uparrow 1.$$

On the other hand, by the Poisson formula,

$$\begin{aligned}\Lambda_R \phi_R &= \int_0^{2\pi} \frac{R^2 - r^2}{1 - 2r\cos(\alpha - \theta) + r^2} u(re^{i\theta})\, d\theta \\ &= 2\pi u(z).\end{aligned}$$

Thus, we have

$$2\pi u(z) = \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\alpha - \theta) + r^2}\, d\mu(\theta),$$

that is, (5.10) holds.

Now, let $g \in C(S^1)$. For a fixed $r < 1$, by Fubini's theorem and Poisson's formula,

$$\begin{aligned}\int_0^{2\pi} u(re^{i\alpha}) g(\alpha)\, d\alpha &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\alpha - \theta) + r^2} g(\alpha)\, d\alpha d\mu(\theta) \\ &= \int_0^{2\pi} h(re^{i\theta})\, d\mu(\theta),\end{aligned}$$

where h is the harmonic function determined by the boundary value g. In case

(5.10) holds for some measure μ_1 instead of μ, the same reasoning shows that

$$\int_0^{2\pi} u(re^{i\alpha})g(\alpha)\,d\alpha = \int_0^{2\pi} h(re^{i\theta})\,d\mu_1(\theta).$$

It follows that

$$\int_0^{2\pi} h(re^{i\theta})\,d\mu_1(\theta) = \int_0^{2\pi} h(re^{i\theta})\,d\mu(\theta).$$

Letting $r \uparrow 1$,

$$\int_0^{2\pi} g(\theta)\,d\mu_1(\theta) = \int_0^{2\pi} g(\theta)\,d\mu(\theta), \quad \forall g \in C(S^1)$$

so $\mu_1 = \mu$. This shows the uniqueness of the boundary trace μ. As every $r_j \uparrow 1$ contains a subsequence $\{r_{j_k}\}$ such that $\Lambda_{r_{kj}}$ converges weakly* to some boundary trace which must be μ, we conclude that (5.9) holds. $\square$

Comments on Chapter 5. Sections 1–4 are taken from [R]. There are different proofs of Radon-Nikodym theorem, check them from the web. See [R] for a proof of $L^p - L^q$ duality based on the representation theorem when the measure is σ-finite. The discussion in Section 4 is parallel to Section 6 in Chapter 4 and one may consult Buttazzo, Gaiquinta and Hildebrandt "One-dimensional Variational Problems" for a complete proof of Dunford-Pettis theorem. Finally, Herglotz-Riesz theorem is taken from Lax "Functional Analysis".