Lecture 8:

Recall: Linear Beltrami Solver (LBS)

Let $M = (V, E, F)$ be simply-connected domain w/ boundary.

Let $V = \{(g_1, h_1), (g_2, h_2), \ldots, (g_{|V|}, h_{|V|})\}$.

In discrete formulation, given $\mu = \rho + i\tau$, we want to compute a resulting mesh M' such that

$$v_n = (g_n, h_n) \mapsto w_n = (s_n, t_n)$$

vertices in M'

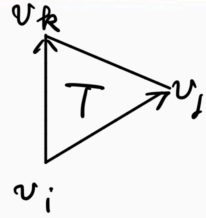
On each face T, the discrete QC map f is linear.

$$\therefore \quad f|_T(x,y) = \begin{pmatrix} u|_T(x,y) \\ v|_T(x,y) \end{pmatrix} = \begin{pmatrix} a_T x + b_T y + r_T \\ c_T x + d_T y + s_T \end{pmatrix}$$

" $u + iv$ "

$$\therefore \quad u_x|_T = a_T \; ; \quad u_y|_T = b_T \; ; \quad v_x|_T = c_T \; ; \quad v_y|_T = d_T$$

Consider the directional derivatives along $v_j - v_i$ and $v_k - v_i$, we get:



$$\begin{pmatrix} a_T & b_T \\ c_T & d_T \end{pmatrix} \begin{pmatrix} g_j - g_i & g_k - g_i \\ h_j - h_i & h_k - h_i \end{pmatrix} = \begin{pmatrix} s_j - s_i & s_k - s_i \\ t_j - t_i & t_k - t_i \end{pmatrix}$$

Assume f is orientation-preserving, then:

$$\det \begin{pmatrix} g_j - g_i & g_k - g_i \\ h_j - h_i & h_k - h_i \end{pmatrix} = 2\,\mathrm{Area}(T).$$

$$\therefore \begin{pmatrix} a_T & b_T \\ c_T & d_T \end{pmatrix} = \frac{1}{2\,\mathrm{Area}(T)} \begin{pmatrix} s_j - s_i & s_k - s_i \\ t_k - t_i & t_k - t_i \end{pmatrix} \begin{pmatrix} h_k - h_i & g_i - g_k \\ h_i - h_j & g_j - g_i \end{pmatrix}$$

$$\begin{pmatrix} a_T & b_T \\ c_T & d_T \end{pmatrix} = \begin{pmatrix} A_T^i s_i + A_T^j s_j + A_T^k s_k & B_T^i s_i + B_T^j s_j + B_T^k s_k \\ A_T^i t_i + A_T^j t_j + A_T^k t_k & B_T^i t_i + B_T^j t_j + B_T^k t_k \end{pmatrix}$$

$$\begin{bmatrix} a_T & b_T \\ c_T & d_T \end{bmatrix} = \frac{1}{2 \cdot Area(T)} \begin{bmatrix} s_j - s_i & s_k - s_i \\ t_j - t_i & t_k - t_i \end{bmatrix} \begin{bmatrix} h_k - h_i & g_i - g_k \\ h_i - h_j & g_j - g_i \end{bmatrix}$$
$$= \begin{bmatrix} A_T^i s_i + A_T^j s_j + A_T^k s_k & B_T^i s_i + B_T^j s_j + B_T^k s_k \\ A_T^i t_i + A_T^j t_j + A_T^k t_k & B_T^i t_i + B_T^j t_j + B_T^k t_k \end{bmatrix}.$$

where

$$A_T^i = (h_j - h_k)/2 \cdot Area(T); \quad A_T^j = (h_k - h_i)/2 \cdot Area(T); \quad A_T^k = (h_i - h_j)/2 \cdot Area(T);$$
$$B_T^i = (g_k - g_j)/2 \cdot Area(T); \quad B_T^j = (g_i - g_k)/2 \cdot Area(T); \quad B_T^k = (g_j - g_i)/2 \cdot Area(T).$$

Now, define : $Div(\underbrace{X_1, X_2}_{\vec{V}=(X_1,X_2)})(v_i) = \sum_{T \in N_i} Area(T) \cdot A_T^i X_1(T) + Area(T) \cdot B_T^i X_2(T)$

$\vec{V} = (X_1, X_2)$ vector field defined on each face T

All faces around v_i

Easy to check: $Div(-d, c) = \sum_{T \in N_i} -Area(T) A_T^i (B_T^i t_i + B_T^j t_j + B_T^k t_k) + Area(T) B_T^i (A_T^i t_i + A_T^j t_j + A_T^k t_k)$

$-d = -v_y$, $c = v_x$

$= 0$

Similarly, $Div(-b, a) = 0$

Recall that:

$$\begin{pmatrix} -v_y \\ v_x \end{pmatrix} = \underbrace{\frac{1}{1-\rho^2-\tau^2}\begin{pmatrix} 1-\rho & -\tau \\ -\tau & \rho+1 \end{pmatrix}\begin{pmatrix} \rho-1 & \tau \\ \tau & -(\rho+1) \end{pmatrix}}_{A}\begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -v_y \\ v_x \end{pmatrix} = A \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

$\therefore$ $\mathrm{Div}\left(A\begin{pmatrix} u_x \\ u_y \end{pmatrix}\right) = 0$ to solve for u with suitable boundary conditions $\left(\Leftrightarrow \mathrm{Div}\left\{A\begin{bmatrix} B_T^i s_i + B_T^j s_j + B_T^k s_k \\ B_T^i t_i + B_T^j t_j + B_T^k t_k \end{bmatrix}\right\} = 0\right)$

(If $M = [0,1]\times[0,1]$, and $M' = [0,1]\times[0,h]$ for some h, we set $u = 0$ on the left boundary, $u = 1$ on the right boundary)

Once u is determined, we can determine

$$h = \sum_T \left(\alpha_T (u_x)_T^2 + 2\beta_T (u_x)_T (u_y)_T + \gamma_T (u_y)_T^2 \right)$$

v can be determined by solving:

$$\mathrm{Div}\left(A \begin{pmatrix} v_x \\ v_y \end{pmatrix} \right) = 0$$

with boundary conditions: $v = 0$ on the bottom boundary

$v = h$ on the upper boundary.

Remark: In case landmark constraints are imposed, we can solve:

$$\mathrm{Div}\left(A\begin{pmatrix} u_x \\ u_y \end{pmatrix}\right) = 0 \quad \text{and} \quad \mathrm{Div}\left(A\begin{pmatrix} v_x \\ v_y \end{pmatrix}\right) = 0$$

subject to $u(p_i) = q_i^u$ and $v(p_i) = q_i^v$ for $i = 1, 2, \ldots, m$

(by substituting them into the linear system)

where $\{p_i\}_{i=1}^m \leftrightarrow \{q_i = q_i^u + i q_i^v\}_{i=1}^m$ denotes the landmark corresponding. It gives a q.c. map whose BC is close to μ. (not exactly same)

$$B\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_j \leftarrow q_j^u \\ \vdots \\ \leftarrow q_\ell^u \\ u_N \end{pmatrix} = 0 \quad \text{etc} \quad \text{and} \quad B'\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} = 0 \quad \left(\begin{array}{l} u_i = \text{value of} \\ u \text{ at} \\ \text{vertex } i \\ \text{etc} \end{array} \right)$$

$\Rightarrow B_1 \vec{u} = \vec{b}_1$ etc ...

Fixing conformality distortion for Fast Spherical Conformal Parameterization

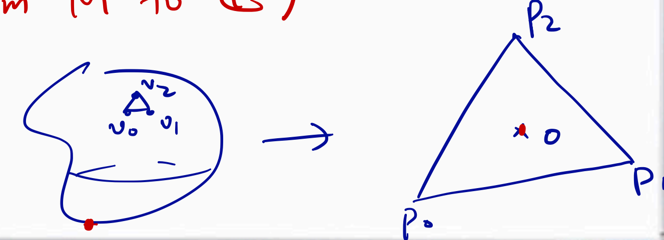
Recall: Given genus-0 mesh $M = (V, E, F)$, we can take away one small triangle $\Delta = [v_0, v_1, v_2]$ (treat it as north pole) and map it to big triangle $T = [p_0, p_1, p_2]$ (w/ same angle structure as Δ) by solving:

$$\sum_{[v_i, v_j] \in E} w_{ij} \left(f(v_j) - f(v_i) \right) = 0$$

subject to the constraint that $f(v_0) = p_0 \in \mathbb{C}$, $f(v_1) = p_1 \in \mathbb{C}$ and $f(v_2) = p_2 \in \mathbb{C}$.

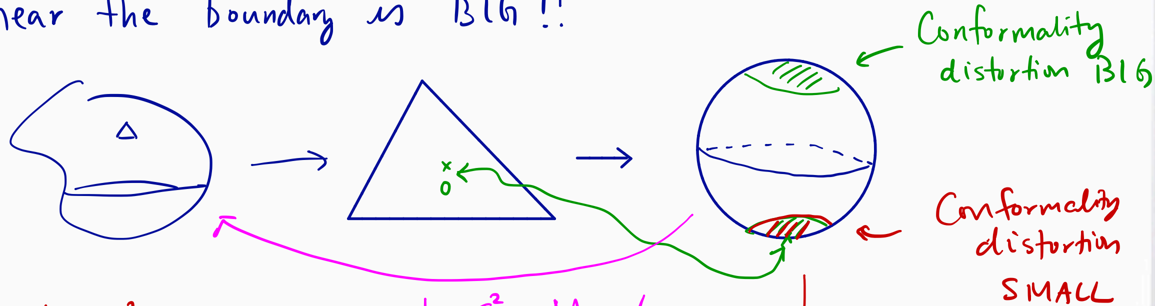
(f is a piecewise linear map from M to $\mathbb{C}$)

(Linear system = fast)



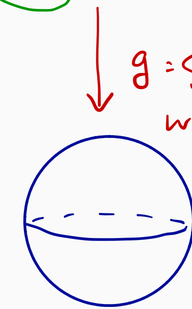
Drawback: Conformality near the origin is small but conformality near the boundary is BIG !!

Strategy:



$\phi : S^2 \to M$ w/ B.C. μ

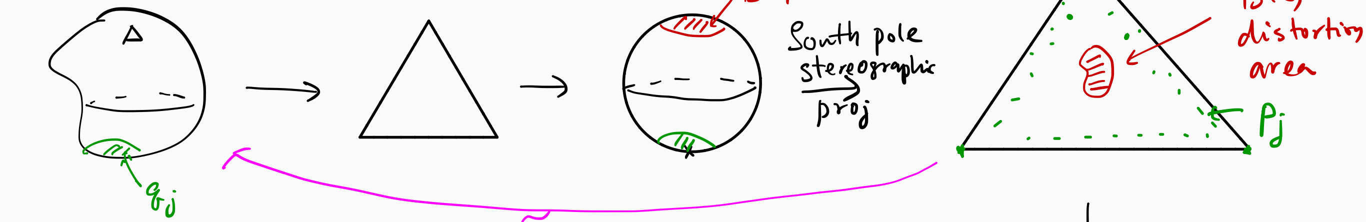
$g : S^2 \to S^2$ w/ B.C. μ



Then: $\phi \circ g^{-1} : S^2 \to M$ has B.C. $= 0$ and hence conformal !!

Computing $g : S^2 \to S^2$ involves conformal chart. Use sterographic projection !!

Detailed computation:



$\tilde{\Phi}$ w/ B.C. = μ

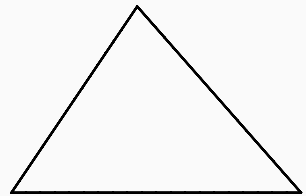
$\tilde{g}$ w/ B.C = μ

" $\tilde{u} + i\tilde{v}$

Solve:

$$\mathrm{Div}\left(A\begin{pmatrix}\tilde{u}_x\\ \tilde{u}_y\end{pmatrix}\right)=0 \quad \text{and} \quad \mathrm{Div}\left(A\begin{pmatrix}\tilde{v}_x\\ \tilde{v}_y\end{pmatrix}\right)=0$$

Subject to $\tilde{g}(p_j) = q_j$ for $j=1,2,$



τ_s^{-1} South pole inverse stereographic proj.

Then: $\tilde{\Phi}\circ\tilde{g}^{-1}\circ\tau_s$ has less conformality distortion near north pole!

Beltrami Holomorphic Flow (Bojarski)

Theorem: (Beltrami Holomorphic flow on $\mathbb{S}^2$) There is a one-to-one correspondence between the set of quasiconformal diffeomorphisms of $\mathbb{S}^2$ that fix the points 0, 1 and ∞ and the set of smooth complex-valued functions μ on $\mathbb{S}^2$ with $\|\mu\|_\infty = k < 1$.

(Here, we identify $\mathbb{S}^2$ with extended complex plane $\overline{\mathbb{C}}$).

Also, the solution to the Beltrami's eqt depends holomorphically on μ. Let $\{\mu(t)\}$ be a family of Beltrami coefficient depending on a real / complex t. Let $\mu(t)(z) = \mu(z) + t\nu(z) + t\varepsilon(t)(z)$ for $z \in \mathbb{C}$ and $\|\varepsilon(t)\|_\infty \to 0$ as $t \to 0$. Then: for $\forall w \in \mathbb{C}$, $\nu, \varepsilon(t) \in L^\infty(\mathbb{C})$

$$f^{\mu(t)}(w) = f^{\mu}(w) + t\,V(f^{\mu}, \nu)(w) + o(|t|) \text{ locally uniformly on } \mathbb{C}$$

as $t \to 0$, where

$$V(f^{\mu}, \nu)(w) = \frac{-f^{\mu}(w)\left(f^{\mu}(w) - 1\right)}{\pi} \int_{\mathbb{C}} \frac{\nu(z)\left((f^{\mu})_z(z)\right)^2}{f^{\mu}(z)\left(f^{\mu}(z) - 1\right)\left(f^{\mu}(z) - f^{\mu}(w)\right)}\, dx\, dy$$

BHF algorithm

Let M = genus-0 mesh
N = genus-0 mesh.

Let $\phi_1 : M \to \mathbb{S}^2 \cong \overline{\mathbb{C}}$ and $\phi_2 : N \to \mathbb{S}^2 \cong \overline{\mathbb{C}}$ be global conformal parameterization of M and N respectively.

Given $\mu : M \to \mathbb{C}$, consider $\tilde{\mu} = \mu \circ \phi_1^{-1} : \mathbb{S}^2 \to \mathbb{C}$.

Goal: Construct $\tilde{g} : \underset{\overline{\mathbb{C}}}{\underset{\shortparallel}{\mathbb{S}^2}} \to \underset{\overline{\mathbb{C}}}{\underset{\shortparallel}{\mathbb{S}^2}} \ni$ B.C. of $\tilde{g} = \tilde{\mu}$

(under "north pole" stereographic proj. as conformal chart)

Identify $\mathbb{S}^2$ with $\overline{\mathbb{C}}$. Define: $\tilde{\mu}_k = k\tilde{\mu}/N$, $k = 0, 1, 2, \ldots N$

Let $\tilde{f}^{\tilde{\mu}_k}$ = Q.C. map associated with $\tilde{\mu}_k$.

$\therefore \tilde{f}^{\tilde{\mu}_0} = Id$ (assuming $0, 1, \infty$ are fixed)

Proceed to construct:

$$\begin{tikzcd}
\tilde{\mu}_0 \arrow[r] \arrow[d, leftrightarrow] & \tilde{\mu}_1 \arrow[r] \arrow[d, leftrightarrow] & \tilde{\mu}_2 \arrow[r] \arrow[d, leftrightarrow] & \cdots \arrow[r] & \tilde{\mu}_k \arrow[r] \arrow[d, leftrightarrow] & \cdots \arrow[r] & \tilde{\mu}_N = \tilde{\mu} \arrow[d, leftrightarrow] \\
\underset{= \mathrm{Id}}{\tilde{f}^{\tilde{\mu}_0}} \arrow[r] & \tilde{f}^{\tilde{\mu}_1} \arrow[r] & \tilde{f}^{\tilde{\mu}_2} \arrow[r] & \cdots \arrow[r] & \tilde{f}^{\tilde{\mu}_k} \arrow[r] & \cdots \arrow[r] & \tilde{f}^{\tilde{\mu}_N} = \tilde{g}
\end{tikzcd}$$

Note that the above diagram can be realized iteratively by:

$$\begin{cases} \tilde{f}^{\tilde{\mu}_0} = \mathrm{Id} \\ \tilde{f}^{\tilde{\mu}_{k+1}} = \tilde{f}^{\tilde{\mu}_k} + \vec{V}\left(\tilde{f}^{\tilde{\mu}_k}, \dfrac{\tilde{\mu}}{N}\right), \end{cases}$$

where $$\vec{V}(f^{\mu}, \nu) = -\int_{\mathbb{C}} \frac{f^{\mu}(w)\left(f^{\mu}(w) - 1\right)}{\pi} \left(\frac{\nu(z)\left((f^{\mu})_z(z)\right)^2}{f^{\mu}(z)\left(f^{\mu}(z) - 1\right)\left(f^{\mu}(z) - f^{\mu}(w)\right)} \right) dx\, dy$$

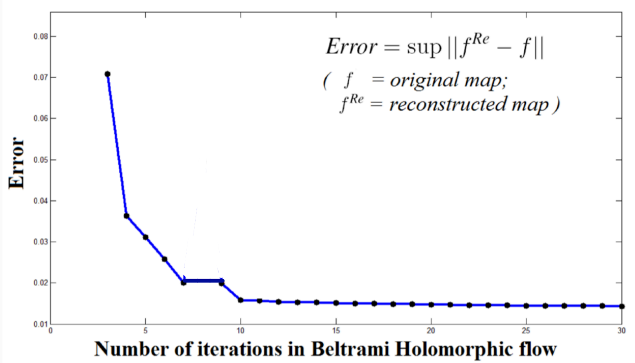
In discrete case, $\vec{V}(f^{\mu}, \nu)$ can be discretized as: $\sum_{v} k(v, w) \overset{(\text{area})}{A_v}$

Algorithm *Reconstruction of Surface Diffeomorphisms from BCs*

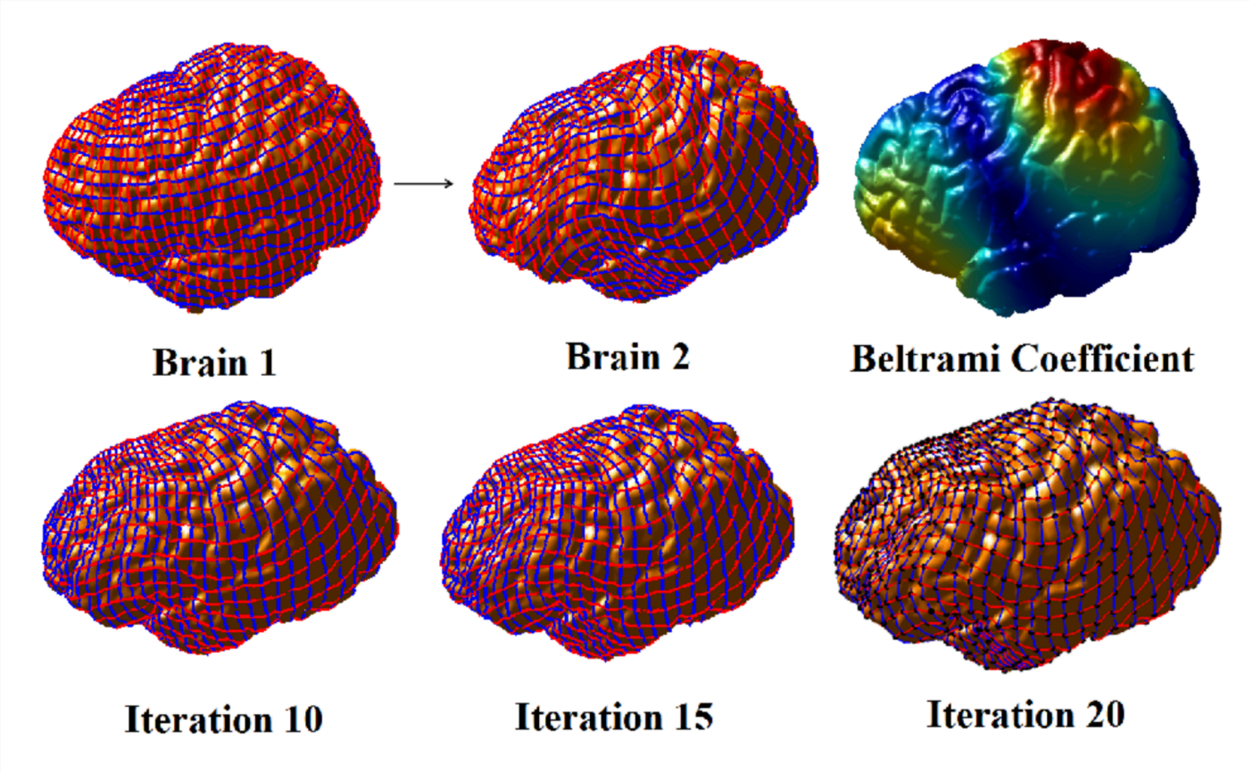
Input: *Beltrami Coefficient* μ *on* S_1; *conformal parameterizations of* S_1 *and* S_2: ϕ_1 *and* ϕ_2; *Number of iterations* N

Output: *Surface diffeomorphism* $f^\mu : S_1 \to S_2$ *associated to* μ.

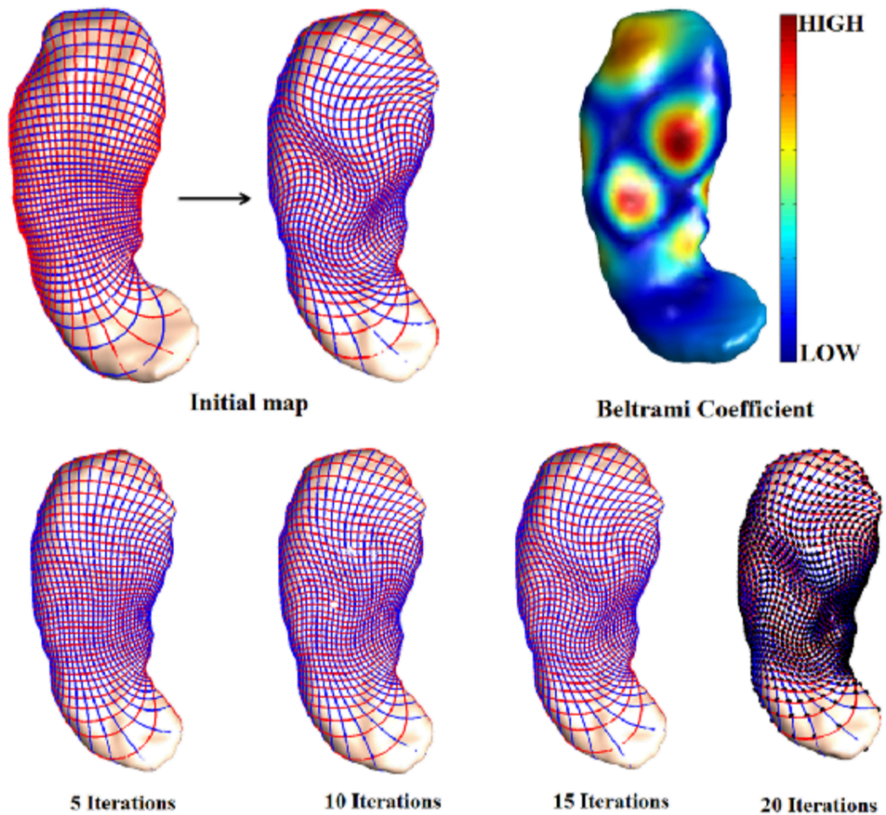
1) *Set* $k = 0$; $\widetilde{f}^{\widetilde{\mu}_0} = \mathbf{Id}$.
2) *Set* $\widetilde{\mu}_k := k\widetilde{\mu}/N$; *Compute* $\widetilde{f}^{\widetilde{\mu}_{k+1}} = \widetilde{f}^{\widetilde{\mu}_k} + V(\widetilde{f}^{\widetilde{\mu}_k}, \frac{\widetilde{\mu}}{N})$; $k = k+1$.
3) *Repeat Step 2 until* $k = N$; *Set* $f^\mu := \phi_2^{-1} \circ \widetilde{f}^{\widetilde{\mu}} \circ \phi_1 : S_1 \to S_2$.



Remark: Can be used to solve optimization problem of mappings represented by Beltrami coefficients.



Brain 1
Brain 2
Beltrami Coefficient
Iteration 10
Iteration 15
Iteration 20


HIGH
LOW
Initial map
Beltrami Coefficient
5 Iterations
10 Iterations
15 Iterations
20 Iterations

We can find the associated QC map $\tilde{f} = f^k + \vec{V}_2 \Rightarrow$

$$\frac{\partial \tilde{f}}{\partial \bar{z}} \approx \tilde{\nu} \frac{\partial \tilde{f}}{\partial z}$$

Overall, the descent direction to optimize:

$$\frac{df^k}{dt} = \vec{V}_1 + \vec{V}_2$$