Computation of disk conformal parameterization

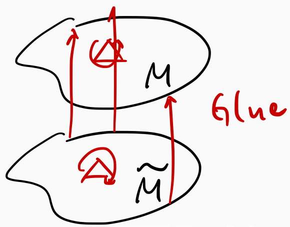
Goal: Given a simply-connected surface S, find $\phi : S \to \mathbb{D}$

Challenge: Cannot get it by computing harmonic map.

(Given a boundary homeomorphism $h : \partial S \to \partial \mathbb{D}$, there exist a unique harmonic map $H : S \to \mathbb{D} \ni H|_{\partial S} = h$. Amongst them, only few of them are conformal w/ suitable boundary conditions)

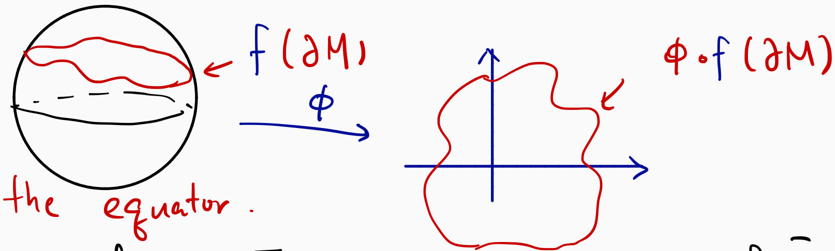
Idea: (Double covering)

- Let $M = (V, E, F)$, construct $\tilde{M} = (V, \tilde{E}, \tilde{F})$ which is a "reflected" copy of M ($\tilde{F}$ has opposite orientation as F).
- Glue them along the boundary.
- $M \cup \tilde{M} = \bar{M}$ becomes a genus-0 closed surface



- $\tilde{M}$ can be parameterized onto $\mathbb{S}^2$ using the previous algorithm.

Problem: $\bar{M} \xrightarrow{f}$



$f(\partial M)$ may not be on the equator.

$\because$ $\bar{M}$ is a symmetric surface, $\exists$ a conformal map $h: \bar{M} \to \bar{\mathbb{C}}$ such that $h(\partial M) = \partial \mathbb{D}$.

Solution: Picking v_0, v_1, v_2 on ∂M. Reparameterize $\phi \circ f$ by a Mobius Transformation τ such that $h = \tau \circ \phi \circ f$ maps v_0, v_1, v_2 to $0, 1, i$ respectively.

e.g. $\underset{\mathbb{C}}{\underset{\cap}{z_0}}, \underset{\mathbb{C}}{\underset{\cap}{z_1}}, \underset{\mathbb{C}}{\underset{\cap}{z_2}}$ to $0, 1, \infty$ can be done by: $\tau_1 = \dfrac{(z-z_0)(z_1-z_2)}{(z-z_2)(z_1-z_0)}$

etc ...

Input: A oriented surface with boundaries M;
Output: The double covering $\bar{M}$;

1. Make a copy of M, denoted as M';
2. Reverse the order of the vertices of each face of M';
3. Glue M and M' along their corresponding boundary edges to obtain $\bar{M}$.

Fast algorithm for genus-0 spherical conformal parameterization

<u>Idea</u>: Let S be a Riemann surface. $\exists$ conformal parameterization $\ni$ $\phi : S \to \mathbb{S}^2$.

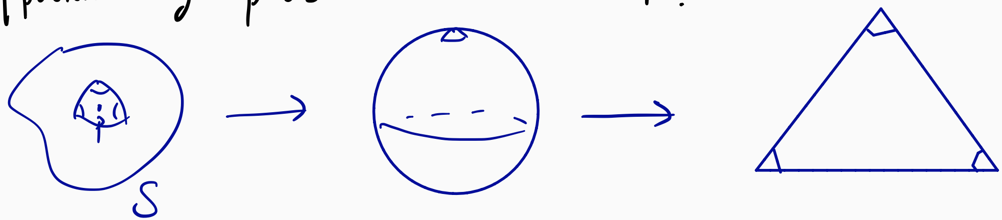
Let $p \in S$. We can assume $\phi(p) =$ north pole.

Let τ be the sterographic projection.

Let Δ be a small "curved" triangle around p $\ni$

$$\underbrace{\tau \circ \phi}_{\tilde{\phi}}(\Delta) = \tilde{\Delta} = \text{big triangle in } \mathbb{C}.$$

The angles at the 3 vertices of Δ is approximately preserved under $\tilde{\phi}$.



Method: In the discrete case, let $M=(V,E,F)$. Take $T=[v_0,v_1,v_2]\in F$ and let $p\in T$ be the centroid of T.

We can find a harmonic map with boundary condition that $\tilde{\phi}(v_0)=w_0$, $\tilde{\phi}(v_1)=w_1$ and $\tilde{\phi}(v_2)=w_2$ $\ni$ $[w_0,w_1,w_2]$ has the same angle structure as $[v_0,v_1,v_2]$.

Mathematically, we need to solve:

$$\sum_{[v_i,v_j]\in E} \omega_{ij}\left(f(v_j)-f(v_i)\right)=0 \quad \text{for } \forall\, i=1,2,\dots,N$$

and fix $f(v_0)=w_0$, $f(v_1)=w_1$ and $f(v_2)=w_2$.

(Linear system, much faster than iterative scheme)

Remark: Numerical error (conformality distortion) near the north pole is big.

We'll use quasiconformal theories to fix it.

Idea: Let $\varphi: S \to \mathbb{S}^2$ with big distortion at north pole.

Reparameterize φ by quasi-conformal map $g \ni$

$g \circ \varphi$ can fix the conformality distortion

Brain landmark matching optimized harmonic parameterization

Goal: Given a brain cortical surface S. Let $\{p_i\}_{i=1}^{N}$ be landmark points defined on S. Want to find: $f: S \to S^2$ such that f is as conformal / harmonic as possible and $f(p_i) = q_i$ $(i=1,2,\dots,m)$ for some fixed locations $q_i \in S^2$.

Suppose S_1 and S_2 be two brain surfaces w/ landmarks $\{p_i\}_{i=1}^{m}$ and $\{p_i'\}_{i=1}^{m}$ respectively. Let $f: S_1 \to S^2$ and $f': S_2 \to S^2$ $\ni$ $f(p_i) = q_i = f'(p_i')$ for $i=1,2,\dots,m$,

Then, $(f')^{-1} \circ f : S_1 \to S_2$ is a landmark-matching surface registration of S_1 and S_2 (Atlas-based surface registration)

<u>Method 1</u>: Find $f \ni \sum_{[v_i,v_j]\in E} w_{ij}\,(f(v_j)-f(v_i)) \quad \forall\, i=1,2,\dots,n$

and $f(p_i)=q_i \quad i=1,2,\dots,m$

(if p_i's are vertices)

<u>Drawback</u>: Bijectivity is difficult to control.

<u>Method 2</u>: Find f that minimizes:

$$E_{landmark}(f) = \frac{1}{2}\sum_{[v_i,v_j]\in E} w_{ij}\,|f(v_i)-f(v_j)|^2 + \lambda\sum_{k=1}^{m}|f(p_k)-q_k|^2$$

λ = adjusting parameter (Big if we want more accurate landmark matching)

Soft constraint can better control bijectivity.

Using same idea, we use descent method to minimize $E_{landmark}$

$$\frac{d\vec{f}}{dt} = -\mathcal{D}\vec{f} \quad , \quad \text{where}$$

$$(\widetilde{\mathcal{D}\vec{f}})_i = \sum_{[v_i,v_j]\in} w_{ij}\,(f(v_j) - f(v_i)) + 2\lambda \sum_{k=1}^{m} (f(p_k) - q_k)$$

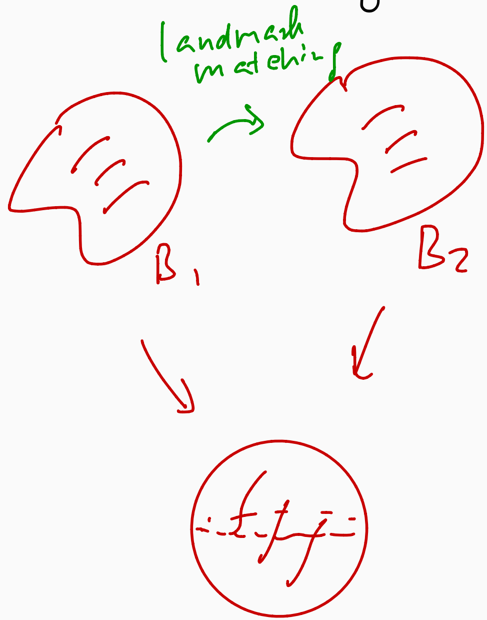
Normalize $(\widetilde{\mathcal{D}\vec{f}})$ to its tangential component to get

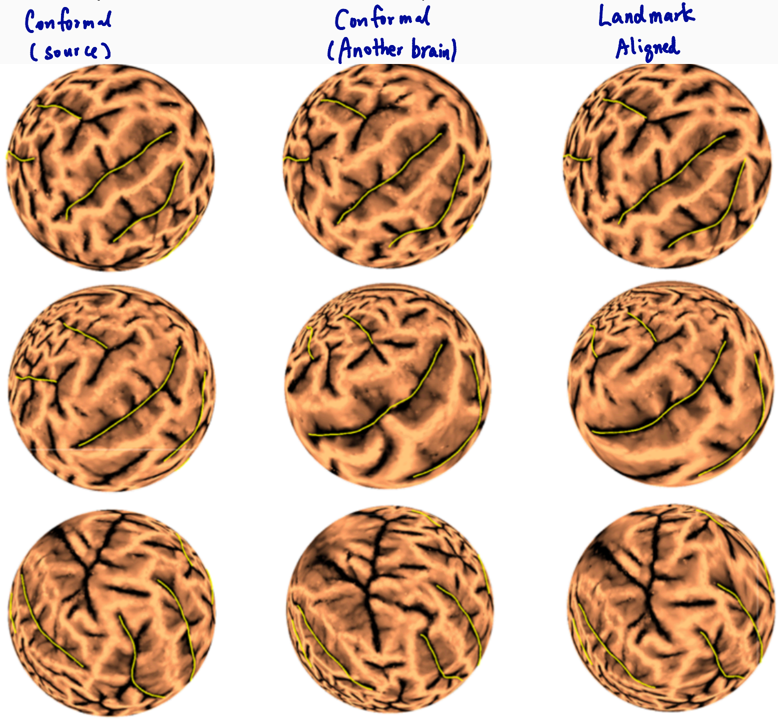
$$\mathcal{D}\vec{f} = (\widetilde{\mathcal{D}\vec{f}}) - \langle \widetilde{\mathcal{D}\vec{f}}, \vec{n} \rangle \vec{n}$$

Iteratively adjust $\vec{f}$ to minimize $E_{landmark}$.

Remark: Both methods do not have bijectivity guarantee.

Use quasiconformal theories to fix it.





Quasiconformal map between Riemann surfaces

Basic idea: Given two Riemann surfaces S_1 and S_2.
Under the conformal coordinate charts, $f: S_1 \to S_2$ is "quasi-conformal" iff f is "quasi-conformal" as a map from $\mathbb{C} \to \mathbb{C}$ under the conformal charts (follows from the definition. Later)

Suppose S_1 and S_2 are simply-connected open surfaces.
$\exists$ conformal $\phi_1 : \mathbb{D} \to S_1$ and $\phi_2 : \mathbb{D} \to S_2$ (Global Conformal parameterization)

Then: $f: S_1 \to S_2$ is quasiconformal iff

$\phi_2^{-1} \circ f \circ \phi_1 : \mathbb{D} \to \mathbb{D}$ is quasi-conformal in 2D.

$\therefore$ Focus our attention on $\mathbb{C} \to \mathbb{C}$ first!

Quasi-conformal map from $\mathbb{C}$ to $\mathbb{C}$

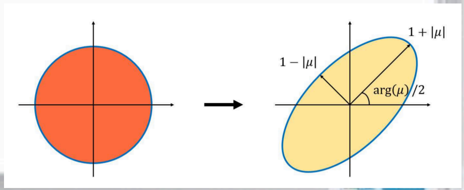
Definition: (Quasiconformal map) Let $f : \mathbb{C} \to \mathbb{C}$ be a C^1 homeomorphism. f is called a quasi-conformal map with respect to a complex-valued function $\mu : \mathbb{C} \to \mathbb{C}$, called the Beltrami coefficient, with $\|\mu\|_\infty < 1$ if:

$$(*) \quad \frac{\partial f}{\partial \bar{z}}(z) = \mu(z) \frac{\partial f}{\partial z} \quad \text{where}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$\mu(z)$ measures the local geometric distortion at z.

$(*)$ is called the Beltrami's equation



Remark: 1. When $\mu \equiv 0$, the Beltrami's equation is reduced to the Cauchy-Riemann equation. Let $f = u + iv$ (u, v real functions).

Then:
$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x}(u+iv) - i\frac{\partial}{\partial y}(u+iv)\right)$$
$$= \frac{1}{2}\left((u_x + v_y) + i(v_x - u_y)\right) = 0$$

$$\Rightarrow \begin{cases} u_x = -v_y \\ u_y = +v_x \end{cases} \quad \text{(Cauchy-Riemann eqt)}$$

2. In matrix form, a conformal/holomorphic complex-value function $f = u + iv$ satisfies:

$$Df(z) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}$$

or $$\begin{pmatrix} -v_y \\ v_x \end{pmatrix} = \overset{\text{Id}}{\begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix}} \begin{pmatrix} u_x \\ u_y \end{pmatrix}. \quad \text{———} \quad (**)$$

Quasi-conformal map generalizes $(**)$ by considering

$$\begin{pmatrix} -v_y \\ v_x \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}}_{\text{Represent the metric distortion}} \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$ for some α, β and γ depending on μ.

3. Let $J(z)$ = Jacobian of $f = u + iv$ at z.

Then $J = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = u_x v_y - u_y v_x$

Note that:

$$\left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2 = \frac{(u_x + v_y)^2 + (v_x - u_y)^2}{4} - \frac{(u_x - v_y)^2 + (v_x + u_y)^2}{4}$$

$$= (u_x v_y - u_y v_x) = J(z)$$

$$\therefore J(z) = \left|\frac{\partial f}{\partial z}\right|^2 \left(1 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2 \Big/ \left|\frac{\partial f}{\partial z}\right|^2\right) = \left|\frac{\partial f}{\partial z}\right|^2 \left(1 - |\mu(z)|^2\right)$$

Thus, if $\|\mu(z)\|_\infty < 1$ and $\left|\frac{\partial f}{\partial z}\right| \neq 0$ (f = homeomorphism)

then $J(z) > 0$ everywhere. $\therefore$ f is orientation-preserving everywhere