Lecture 4:

Recap: Let $\mathcal{M}$ be a smooth surface.

- A Riemannian metric g associated to $\mathcal{M}$ is defined:

For $\forall p \in \mathcal{M}$, $g_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ defines an inner product $\ni$ $\underbrace{\langle \vec{v}, \vec{w} \rangle}_{\text{inner product}} = g_p(\vec{v}, \vec{w})$ for all $\vec{v}, w \in T_p\mathcal{M}$

(From Linear Algebra, at each $p \in \mathcal{M}$, g_p is associated to a 2×2 SPD matrix $\begin{pmatrix} g_{11}(p) & g_{12}(p) \\ g_{21}(p) & g_{22}(p) \end{pmatrix}$ and in every smooth local coordinates (x^1, x^2),

$$g_p = \sum_{i,j=1}^{2} g_{ij}(p)\, dx^i\, dx^j .$$

g_{ij}'s are smooth)

- Any metric surface M is associated to an isothermal coordinates.

  That's, let $\{(U_\alpha, z_\alpha)\}_{\alpha\in A}$ be the conformal atlas for M. ($z_\alpha = x_\alpha + i y_\alpha$)

  Then: $$g = e^{\lambda(z_\alpha)}\left(dx_\alpha^2 + dy_\alpha^2\right)$$

- Any metric surface is a Riemann surface.

- $$\begin{tikzcd} S_1 \arrow[r, "f"] \arrow[d, "z_\alpha"] & S_2 \arrow[d, "w_\beta"] \\ U_\alpha \arrow[r] & V_\beta \end{tikzcd}$$

  $\tilde{f} = w_\beta \circ f \circ z_\alpha^{-1}$

  f is conformal iff $\tilde{f}$ is conformal for all z_α, w_β

Basic theories of planar conformal maps

Theorem: (Riemann mapping) Suppose $D \subset \mathbb{C}$ is a simply-connected domain on the complex plane, the boundary ∂D has more than one point, $z_0 \in D$ is an arbitrary interior point. Then, there exists a unique conformal mapping $\phi : D \to \Delta$ from D to the unit disk Δ, such that $\phi(z_0) = 0$ and $\phi'(z_0) > 0$.

Remark: If $f : S \to \mathbb{D}$ and $g : S \to \mathbb{D}$ are disk conformal parameterization of S, then: $g \circ f^{-1}$ is a conformal map between unit disk

$$\begin{array}{ccc} S & & S \\ f \downarrow & & \downarrow g \\ \mathbb{D} & \longrightarrow & \mathbb{D} \end{array}$$

$\therefore \quad g \circ f^{-1}(z) = \underbrace{e^{i\theta}\left(\frac{z-a}{1-\bar{a}z}\right)}_{\phi(z)}$ for some $\underset{\mathbb{C}}{a}, \underset{[0,2\pi)}{\theta}$

$\therefore \quad \boxed{g = f \circ \phi}$

Surface harmonic map : theories and computation

Basic theoretical background

1. Let $f : M \to \mathbb{R}$. The differential of f is defined as:

$$df_p(\vec{v}) \overset{def}{:=} D_{\vec{v}} f \quad \text{for } \forall \vec{v} \in T_pM$$

$$= \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) \quad \text{where} \quad \left.\frac{d}{dt}\right|_{t=0} \gamma(t) = \vec{v}$$

Under the coordinate chart (x^1, x^2) around p,

$$df_p := \sum_{i=1}^{2} \frac{\partial f}{\partial x^i}(p)\, dx^i \qquad \begin{pmatrix} dx^1((v^1, v^2)) = v_1 \\ dx^2((v^1, v^2)) = v_2 \end{pmatrix}$$

2. (Planar harmonic function) Let $\Omega \subseteq \mathbb{R}^2$ and let $u : \Omega \to \mathbb{R}$.

u is said to be a harmonic function if : $\Delta u = 0$

Harmonic map and energy minimization

Consider: $$E(u) \stackrel{\text{def}}{=} \int_\Omega \langle \nabla u, \nabla u \rangle \, dx\,dy$$

Suppose u minimizes $E(u)$, then:

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(u+\varepsilon h) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_\Omega \langle \nabla u + \varepsilon \nabla h, \nabla u + \varepsilon \nabla h \rangle \, dx\,dy$$

$$= 2\int_\Omega \langle \nabla u, \nabla h \rangle \, dx\,dy$$

Fixing the boundary, we have $h \equiv 0$ on $\partial\Omega$.

Integration by part gives: $0 = 2\int_\Omega h \Delta u \, dx\,dy$ for $\forall h$, $h|_{\partial\Omega} \equiv 0$

$$\therefore \quad \begin{cases} \Delta u \equiv 0 \\ u|_{\partial\Omega} = g \quad \text{(Boundary condition)} \end{cases}$$

Remark: • A harmonic function minimizes the harmonic energy $E(u) = \int_\Omega \langle \nabla u, \nabla u \rangle \, dx\, dy$

• A map $f : \Omega \subseteq \mathbb{R}^2 \to \Omega' \subseteq \mathbb{R}^2$ is said to be <u>harmonic</u> if $f \overset{\text{def}}{:=} u + iv$, $\Delta u \equiv 0$ and $\Delta v \equiv 0$.

• A map $f : S \to \Omega \subseteq \mathbb{R}^2$ (where S is a Riemann surface) is a harmonic map if with respect to a (conformal) coordinate chart ϕ, $f \circ \phi$ is a harmonic map.

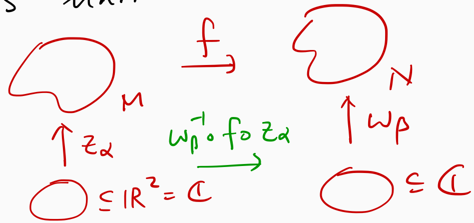
How about harmonic energy between Riemann surfaces?

Consider $f : (M, g) \to (N, h)$ where g and h are Riemannian metric on M and N respectively.

Under the isothermal coordinates (U_α, z_α) and (V_β, w_β),

f is harmonic iff $w_\beta^{-1} \circ f \circ z_\alpha$ is harmonic

iff $w_\beta^{-1} \circ f \circ z_\alpha$ minimizes harmonic energy.



Definition: The homeomorphism $f : M \to N$ is a harmonic map if f minimizes the harmonic energy.

Computation of discrete harmonic map

Let M be a triangulated surface. A piecewise linear function or map is a function/map on M such that it is linear on each triangular face.

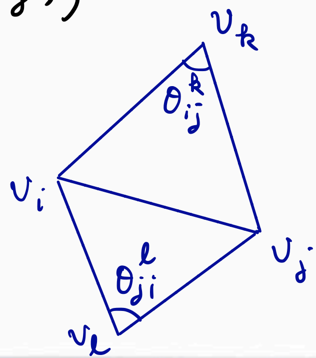
<u>Theorem</u>: Given a piecewise linear function $f: M \to \mathbb{R}$, then the harmonic energy of f is given by:

$$E(f) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij} \left(f(v_i) - f(v_j)\right)^2$$

where

$$w_{ij} = \cot \theta_{ij}^{k} + \cot \theta_{ji}^{l}$$

(Cotangent formula)



Definition: (Bary-centric coordinates)

Given a Euclidean triangle with v_i, v_j, v_k, the bary-centric coordinates of a planar point $p \in \mathbb{R}^2$ with respect to the triangle are $(\lambda_i, \lambda_j, \lambda_k)$, $p = \lambda_i v_i + \lambda_j v_j + \lambda_k v_k$, where

$$\lambda_i = \frac{(v_j - p) \times (v_k - v_j) \cdot \vec{n}}{(v_j - v_i) \times (v_k - v_i) \cdot \vec{n}}$$

λ_j, λ_k are defined similarly.

Remark:

- $\lambda_i + \lambda_j + \lambda_k = 1$ (Check)
- If p is the interior point of the triangle, then all components of the bary-centric coordinates are positive.

Lemma: Suppose $f: \Delta^{\subseteq \mathbb{R}^2} \to \mathbb{R}$ is a linear function,

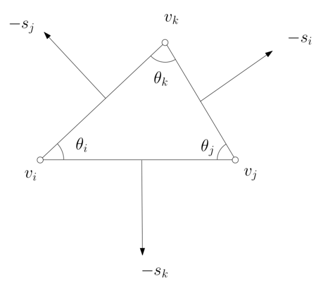
$$f(p) = \lambda_i f(v_i) + \lambda_j f(v_j) + \lambda_k f(v_k),$$

then the gradient of f is: (A = area of Δ)

$$\nabla f(p) = \frac{1}{2A}(s_i f(v_i) + s_j f(v_j) + s_k f(v_k)).$$

Thus, the harmonic energy on a triangle Δ is given by:

$$\int_\Delta \langle \nabla f, \nabla f \rangle dA = \frac{\cot\theta_i}{2}(f_j - f_k)^2 + \frac{\cot\theta_j}{2}(f_k - f_i)^2 + \frac{\cot\theta_k}{2}(f_i - f_j)^2$$



Proof: Note that:

$$S_i + S_j + S_k = n \times \{(v_k - v_j) + (v_i - v_k) + (v_j - v_i)\} = \vec{0}$$

$$\therefore \langle S_i, S_i \rangle = \langle S_i, -S_j - S_k \rangle = -\langle S_i, S_j \rangle - \langle S_i, S_k \rangle.$$

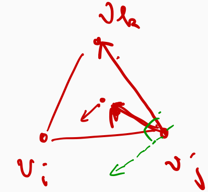
Pick a point $p = \lambda_i v_i + \lambda_j v_j + \lambda_k v_k$. The bary-centric coordinates are given by:

$$\lambda_i = \frac{1}{2A} \langle \underbrace{(v_k - v_j) \times (p - v_j)}_{|v_k - v_j|\,|p - v_j| \sin\theta}, \vec{n} \rangle = \frac{1}{2A} \langle \vec{n} \times (v_k - v_j), p - v_j \rangle$$

$|v_k - v_j|\,|p - v_j| \cos(90 - \theta)$

$$\therefore \lambda_i = \frac{1}{2A} \langle p - v_j, S_i \rangle, \quad \lambda_j = \frac{1}{2A} \langle p - v_k, S_j \rangle$$

$$\lambda_k = \frac{1}{2A} \langle p - v_i, S_k \rangle$$



where A is the triangle area.

$$\therefore \quad f(p) = \lambda_i f_i + \lambda_j f_j + \lambda_k f_k$$

$$= \frac{1}{2A} \langle p - v_j, f_i s_i \rangle + \frac{1}{2A} \langle p - v_k, f_j s_j \rangle + \frac{1}{2A} \langle p - v_i, f_k s_k \rangle$$

$$= \langle p, \frac{1}{2A} (f_i s_i + f_j s_j + f_k s_k) \rangle - \frac{1}{2A} \Big(\langle v_j, f_i s_i \rangle + \langle v_k, f_j s_j \rangle + \langle v_i, f_k s_k \rangle \Big)$$

Hence, $\quad \nabla f = \frac{1}{2A} (f_i s_i + f_j s_j + f_k s_k)$

$$\therefore \int_\Delta \langle \nabla f, \nabla f \rangle \, dA = \frac{1}{4A} \langle f_i s_i + f_j s_j + f_k s_k , f_i s_i + f_j s_j + f_k s_k \rangle$$

(Using the fact that $\langle s_i, s_i \rangle = - \langle s_i, s_j + s_k \rangle$ etc, we can obtain:)

$$= -\frac{1}{4A} \Big(\langle s_i, s_j \rangle (f_i - f_j)^2 + \langle s_j, s_k \rangle (f_j - f_k)^2 + \langle s_k, s_i \rangle (f_k - f_i)^2 \Big)$$

$\because \quad \dfrac{\langle s_i, s_j \rangle}{2A} = -\cot\theta_k, \quad \dfrac{\langle s_j, s_k \rangle}{2A} = -\cot\theta_i, \quad \dfrac{\langle s_k, s_i \rangle}{2A} = -\cot\theta_j$

$$\therefore \int_\Delta \langle \nabla f, \nabla f \rangle \, dA = \frac{\cot\theta_i}{2}(f_j - f_k)^2 + \frac{\cot\theta_j}{2}(f_k - f_i)^2 + \frac{\cot\theta_k}{2}(f_i - f_j)^2$$

<u>Remark</u>: • Let $f: M \to \Omega \subseteq \mathbb{R}^2$ be a discrete map between triangular meshes. Then, each triangle $\Delta \subset M$ can be flatten in $\mathbb{R}^2$.

The harmonic energy on each triangle = harmonic energy from flatten triangle to Ω.

• Adding the harmonic energies on all faces together, and merge items associated with the same edge, then each edge contributes $\frac{1}{2} w_{ij} (f_j - f_i)^2$ where

$$w_{ij} = \cot\theta_{ij}^{k} + \cot\theta_{ji}^{l}$$

Definition: (Laplace operator) The discrete Laplacian Δ_{PL} on a piecewise linear function f is

$$\Delta_{PL} f(v_i) = \sum_{[v_i, v_j] \in M} w_{ij} (f(v_j) - f(v_i))$$

Hence, if f minimizes the discrete harmonic energy, then:

$$\Delta_{PL} f \equiv 0$$

Remark: The motivation of this definition is by taking the derivative of the discrete harmonic energy:

$$E(f) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij} (f(v_j) - f(v_i))^2$$

Recall: The Euler-Lagrange eq^t of $\int_M |\nabla f|^2$ is given by $\Delta f = 0$.

Computational Algorithm for Disk Harmonic Maps

Input:A topological disk M;

Output:A harmonic map $\varphi : M \to \mathbb{D}^2$

1. Construct boundary map to the unit circle, $g : \partial M \to \mathbb{S}^1$, g should be a homeomorphism;
2. Compute the cotangent edge weight;
3. for each interior vertex $v_i \in M$, compute Laplacian

$$\Delta\varphi(v_i) = \sum_{v_j \sim v_i} w_{ij}(\varphi(v_i) - \varphi(v_j)) = 0;$$

4. Solve the linear system, to obtain φ.

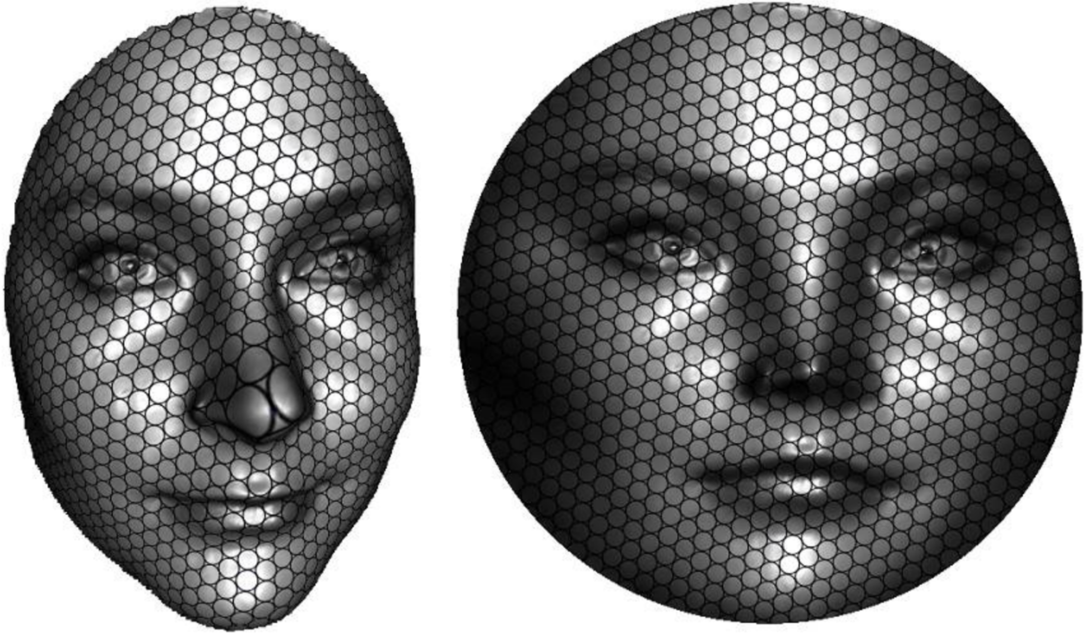


Figure: Harmonic map between topological disks.